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Hybrid Finite Difference Fifth-Order Multi-Resolution WENO Scheme for Hamilton-Jacobi Equations

Zhenming Wang[1], Jun Zhu[2], Linlin Tian[3] and Ning Zhao[4,*]

[1] *School of Mathematics, Key Laboratory of Mathematical Modelling and High Performance Computing of Air Vehicles, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu 210016, P.R. China.*
[2] *State Key Laboratory of Mechanics and Control for Aerospace Structures, Key Laboratory of Mathematical Modelling and High Performance Computing of Air Vehicles, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu 210016, P.R. China.*
[3] *Jiangsu Key Laboratory of Hi-Tech Research for Wind Turbine Design, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu 210016, P.R. China.*
[4] *College of Aerospace Engineering, State Key Laboratory of Mechanics and Control for Aerospace Structures, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu 210016, P.R. China.*

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Abstract. In this paper, a fifth-order hybrid multi-resolution weighted essentially non-oscillatory (WENO) scheme in the finite difference framework is proposed for solving one- and two-dimensional Hamilton-Jacobi equations. Firstly, a new discontinuity sensor is designed based on the extreme values of the highest degree polynomial in the multi-resolution WENO procedures. This hybrid strategy does not contain any human parameters related to specific problems and can identify the troubled grid points accurately and automatically. Secondly, a hybrid multi-resolution WENO scheme for Hamilton-Jacobi equations is developed based on the above discontinuity sensor and a simplified multi-resolution WENO scheme, which yields uniform high-order accuracy in smooth regions and sharply resolves discontinuities. Compared with the existing multi-resolution WENO scheme, the method developed in this paper can inherit its many advantages and is more efficient. Finally, some benchmark numerical experiments are performed to demonstrate the performance of the presented fifth-order hybrid multi-resolution WENO scheme for one- and two-dimensional Hamilton-Jacobi equations.

AMS subject classifications: 65M06
Key words: Discontinuity sensor, Hamilton-Jacobi equations, hybrid method, high order accuracy, multi-resolution WENO scheme.

*Corresponding author. *Email addresses:* wangzhenming@nuaa.edu.cn (Z. Wang), zhujun@nuaa.edu.cn (J. Zhu), lins_tian@nuaa.edu.cn (L. Tian), zhaoam@nuaa.edu.cn (N. Zhao)

1 Introduction

In this paper, a new fifth-order hybrid multi-resolution weighted essentially non-oscillatory (WENO) scheme in finite difference framework is developed for the following Hamilton-Jacobi (H-J) equations:

$$\begin{cases} \phi_t + H(x_1,\cdots,x_n,t,\phi,\nabla\phi) = 0, & (x_1,\cdots,x_n,t) \in \Omega \times [0,\infty), \\ \phi(x_1,\cdots,x_n,0) = \phi_0(x_1,\cdots,x_n), & (x_1,\cdots,x_n) \in \Omega, \end{cases} \tag{1.1}$$

where $\nabla\phi = (\phi_{x_1},\cdots,\phi_{x_n})^T$. As we all know, the H-J equations involve many application fields, such as geometric optics [2], calculus of variations [4], optimal control theory [20], differential games [23], computer graphics [27], image processing [29], and so on [30]. In addition, the solutions to (1.1) typically are Lipschitz continuous but may contain discontinuous derivatives, even with smooth initial conditions. This feature puts forward higher requirements for the numerical algorithms [1,8]. Therefore, many scholars devote their energy to the numerical algorithm research of H-J equations.

The H-J equations are closely related to the hyperbolic conservation laws (such as the Euler equations), so the ideas of numerical methods for hyperbolic conservation laws can be borrowed to solve the H-J equations. In 1991, Osher and Shu generalized the essentially non-oscillatory (ENO) scheme for conservation laws to H-J equations in [23]. And it is extended to unstructured grids by Augoula and Abgrall in [2]. Cecil et al. utilize the radial basis function to construct the ENO scheme for H-J equations in [6]. Compared with the ENO scheme, the research on WENO-type schemes for H-J equations is more abundant. The general framework of the WENO [12] scheme with arbitrary order accuracy was first given by Jiang and Shu in 1996 and was used to solve hyperbolic conservation laws. Subsequently, Jiang and Peng extended it to the solution of the H-J equations in [11]. On this basis, many scholars have carried out a lot of research on WENO schemes for H-J equations. For example, Zhang and Shu extended the work of [11] to the unstructured mesh in [34]. Bryson and Levy presented a central WENO scheme in [5] and extended it to the triangular meshes [15]. In [24], Qiu proposed a WENO scheme with Lax-Wendroff type time discretizations. And a series of work on Hermite WENO (HWENO) methods for solving HJ equations [21,25,36–38]. Recently, a series of unequal-sized WENO schemes for the H-J equations have been proposed in [18, 26, 39]. There is also some other work on WENO schemes to solve H-J equations, such as [14, 19, 28], which will not be introduced here. In addition, there are also some works about solving H-J equations by the high-order discontinuous Galerkin (DG) method. For example, Hu and Shu [10] presented an unstructured DG method for H-J equations in 1999. Cheng and Shu [7] proposed a DG method for solving H-J equations approximating directly the solution variable rather than its derivatives as in [10]. In 2004, Li and Shu [16] reinterpret a DG method originally developed in [10] for H-J equations, which automatically satisfy the curl-free property of the exact solutions inside each element. A new local DG method is proposed by Yan and Osher in [33] and a central DG method is developed in [17]. For