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INSTABILITY OF CRANK-NICOLSON LEAP-FROG FOR NONAUTONOMOUS SYSTEMS

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Abstract. The implicit-explicit combination of Crank-Nicolson and Leap-Frog methods is widely used for atmosphere, ocean and climate simulations. Its stability under a CFL condition in the autonomous case was proven by Fourier methods in 1962 and by energy methods for autonomous systems in 2012. We provide an energy estimate showing that solution energy can grow with time in the nonautonomous case, with worst case rate proportional to time step size. We present two constructions showing that this worst case growth rate is attained for a sequence of timesteps $\triangle t \to 0$. The construction exhibiting this growth for leapfrog is for a problem with a periodic coefficient.

Key words. partitioned methods, energy stability.

1. Introduction

Stability of CNLF, the Crank-Nicolson Leap-Frog method (CNLF) below, is considered for systems with *nonautonomous* $A(t), \Lambda(t)$:

$$\frac{du}{dt} + A(t)u + \Lambda(t)u = 0, \quad \text{for } t > 0, \text{ and } u(0) = u_0. \tag{1.1}$$

Here $A(t), \Lambda(t)$ are $d \times d$ matrices and $u(t)$ is a d vector. $A(t)$ is positive semi-definite symmetric and $\Lambda(t)$ is skew symmetric. Let $|\cdot|_2$ denote the euclidean norm. The CNLF discretization of (1.1) is expressed as follows. Let $t^n = n\Delta t$; given u^0, u^1 find $u^n \in X$ for $n \geq 2$ satisfying

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} + A(t^n)\frac{u^{n+1} + u^{n-1}}{2} + \Lambda(t^n)u^n = 0, \tag{CNLF}$$

with approximations to appropriate accuracy, [23], at the first two time steps. CNLF is the implicit-explicit (IMEX) method used for the dynamic core of most current atmosphere, ocean and climate codes, e.g., [6], [15], [22], [13] and other geophysics problems, [16] .

Stability was shown for the *scalar, autonomous* case under the timestep condition

$$\Delta t|\Lambda|_2 \leq \alpha < 1, \tag{1.2}$$

in 1963 by Johansson and Kreiss [14] and for (non-commuting) autonomous systems in 2012 [17], see also [23], [6] for background. We prove herein *weak instability* in the nonautonomous case.

Remark 1. *CNLF is often used in geophysical fluid dynamics codes together with a time filter. The general strategy used is to split the (nonlinear) equations of motion into terms corresponding to high speed/low energy waves and low speed/high energy waves. The respective terms are discretized by CN and LF with time filters. Williams* [24]*, for example, lists 20 atmosphere codes, 15 ocean codes and 24 coupled / geophysics codes based on this approach. (The precise realization varies with*

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the effects included and the implementation. For one detailed development of one such splitting see Section 3 in [9]*.) Linearization of the split system leads to nonautonomous systems of the form (1.1) above. Time filters are used with CNLF, e.g.,* [3], [19], [24], [13], *with the usual explanation that the latter controls the unstable mode of CNLF. However, the (so-called) unstable mode (or computational mode) of CNLF has recently been proven in* [12] *to be asymptotically stable under (1.2). We do not study time filters herein. However, the instability result does suggest that one positive contribution of time filters may be to control the weak instability identified herein that arises from nonautonomous and periodic* $\Lambda(t)$*.*

The extension of stability theory for linear multistep methods from autonomous to nonautonomous (with test problem $y' = \lambda(t)y$) has a rich history. Dahlquist [5] proved that an A-stable method is similarly stable for $y' = \lambda(t)y$ when $Re\,(\lambda(t)) \leq 0$, further developed in [18]. For the corresponding AN-stability theory for Runge-Kutta methods, see Hundsdorfer and Stetter [10]. For non-A-stable multi-step methods, nonautonomous stability theory was recently developed in [4] with both stability and instability conditions for $y' = \lambda(t)y$. Given a linear multistep method for $y' = \lambda(t)y$, let $\rho(z), \sigma(z)$ be the complex polynomials associated with the method in a standard way and form

$$\mathcal{A} := Re\left[\frac{\rho(z)}{\sigma(z)}\right]_{z=i}.$$

Even if Δt is small enough to be in the stability region of the method, if $\mathcal{A} < 0$ then there exists a $\lambda(t) < 0$ for which the method is unstable [4].

While the theory in [4] does not apply to IMEX methods like CNLF, it can be applied to the special case $A(t) = 0$. The leapfrog method ($A(t) = 0$) is an important wedge example. Indeed, for leapfrog, we calculate $\rho(z) = \frac{1}{2}z^2 - \frac{1}{2}$, and $\sigma(z) = z$. Thus $\mathcal{A} = 0$ and the theory of [4] is inconclusive. Many interesting behaviors are possible between exponential asymptotic stability and exponential instability. One hint is that there is a rich catalog (e.g., [1],[20],[21],[25]) of exotic behavior of leapfrog for Burgers equation starting (to our knowledge) with Fornberg's 1973 paper [8].

The results are clearest for the case that $\Lambda(t)$ is Lipschitz,

$$|\Lambda(t^n) - \Lambda(t^{n-1})|_2 \leq a_0 \Delta t. \tag{1.3}$$

We prove in Theorem 1 that any instability is, at worst, a weak one:

$$|u^{N+1}|_2^2 + |u^N|_2^2 \leq C(\alpha, u^0, u^1) \exp\left[\Delta t \frac{a_0}{1-\alpha} t^N\right]. \tag{1.4}$$

The rate constant $\Delta t a_0/(1-\alpha) \to 0$ as $\Delta t \to 0$ but $\exp\left[\Delta t \frac{a_0}{1-\alpha} t^N\right] \to \infty$ as $t^N \to \infty$. However, the true solution of (1.1) is uniformly bounded and if $A(t) > 0$, $u(t) \to 0$ as $t \to \infty$. Section 3 presents give two constructions that show that (1.4) is best possible for the leapfrog case ($A(t) = 0$).

Remark 2. *Theorem 2.5 in* [4] *shows that for* $\mathcal{A} < 0$*, then there is an alternating pair of states that together lead to growth. The first construction in Section 3 shows similar behavior for* $\mathcal{A} = 0$*.*

In Section 3 we give two constructions that show that LF (and thus CNLF) is exponentially unstable *for arbitrarily small timesteps* when $\Lambda(t)$ is a bounded function that changes sign periodically. A numerical study is also given in Section 3. The numerical data suggest that the instability occurs for a sparse set of