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MONOTONE RELAXATION ITERATES AND APPLICATIONS TO SEMILINEAR SINGULARLY PERTURBED PROBLEMS

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Abstract. This paper deals with monotone relaxation iterates for solving nonlinear monotone difference schemes of elliptic type. The monotone ω-Jacobi and SUR (Successive Under-Relaxation) methods are constructed. The monotone methods solve only linear discrete systems at each iterative step and converge monotonically to the exact solution of the nonlinear monotone difference schemes. Convergent rates of the monotone methods are estimated. The proposed methods are applied to solving semilinear singularly perturbed reaction-diffusion problems. Uniform convergence of the monotone methods is proved. Numerical experiments complement the theoretical results.

Key words. semilinear elliptic problem, monotone difference schemes, monotone relaxation iterates, singularly perturbed problems, uniform convergence.

1. Introduction

Difference schemes, which satisfy the maximum principle, are said to be monotone. The monotonicity condition guarantees that systems of algebraic equations based on such difference schemes are well-posed.

A major point about the nonlinear monotone difference schemes is to obtain reliable and efficient computational methods for computing the solution. The reliability of iterative techniques for solving nonlinear difference schemes can be essentially improved by using componentwise monotone globally convergent iterations. Such methods can be controlled every time. A fruitful method for the treatment of these nonlinear schemes is the method of upper and lower solutions and its associated monotone iterations [5]. Since an initial iteration in the monotone iterative method is either an upper or lower solution, which can be constructed directly from the difference equation without any knowledge of the exact solution, this method simplifies the search for the initial iteration as is often required in the Newton method. In the context of solving systems of nonlinear equations, the monotone iterative method belongs to the class of methods based on convergence under partial ordering (see Chapter 13 in [5] for details).

The purpose of this paper is to extend the monotone iterative method from [3] to monotone relaxation methods of Jacobi- and Gauss–Seidel-type iterations for solving nonlinear monotone difference schemes in the canonical form.

The structure of the paper is as follows. In Section 2, we present the nonlinear monotone difference schemes in the canonical form and formulate the maximum principle. In Section 3, we construct the monotone ω-Jacobi and SUR (Successive Under-Relaxation) methods and prove their monotone convergence. Section 4 is devoted to estimation of convergent rates of the monotone methods. In the final Section 5, the monotone methods are applied to solving singularly perturbed reaction-diffusion problems. We prove that on layer-adapted meshes the monotone

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methods converge uniformly in perturbation parameters. Numerical experiments complement the theoretical results.

2. A nonlinear difference scheme

Let $\overline{\Omega}$ be a bounded computational domain in $\mathbb{R}^k$ $(k = 1, 2, \ldots)$ and $\overline{\Omega}^h$ be a corresponding mesh. For a k-dimensional mesh function $v(p)$, $p \in \overline{\Omega}^h$, consider the nonlinear difference scheme in the canonical form [6]

$$\mathcal{L}v(p) + f(p, v) = 0, \quad p \in \Omega^h, \quad v(p) = g(p), \quad p \in \partial\Omega^h, \tag{1}$$

$$\mathcal{L}v(p) \equiv d(p)v(p) - \sum_{p' \in \sigma'(p)} e(p, p')v(p'),$$

where $\overline{\Omega}^h = \Omega^h \cup \partial\Omega^h$, $\sigma'(p) = \sigma(p) \setminus \{p\}$, $\sigma(p)$ is a stencil of the scheme at an interior mesh point $p \in \Omega^h$ and $\partial\Omega^h$ is the boundary of $\overline{\Omega}^h$. We assume that f is a smooth function, and make the following assumptions on the coefficients of the difference operator $\mathcal{L}$:

$$d(p) > 0, \quad e(p, p') \geq 0, \quad d(p) - \sum_{p' \in \sigma'(p)} e(p, p') \geq 0, \quad p \in \Omega^h. \tag{2}$$

We also assume that the mesh $\overline{\Omega}^h$ is connected. It means that for two interior mesh points $\tilde{p}$ and $\hat{p}$, there exists a finite set of interior mesh points $\{p_1, p_2, \ldots, p_t\}$ such that

$$p_1 \in \sigma'(\tilde{p}), \quad p_2 \in \sigma'(p_1), \ldots, p_t \in \sigma'(p_{t-1}), \quad \hat{p} \in \sigma'(p_t). \tag{3}$$

Introduce the linear version of problem (1)

$$(\mathcal{L} + c)w(p) = f_0(p), \quad p \in \Omega^h, \tag{4}$$

$$w(p) = g(p), \quad p \in \partial\Omega^h, \quad c(p) \geq 0, \quad p \in \overline{\Omega}^h.$$

We now formulate a discrete maximum principle for the difference operator $\mathcal{L} + c$.

Lemma 1. *Let the coefficients of the difference operator $\mathcal{L}$ from (4) satisfy (2) and the mesh $\overline{\Omega}^h$ be connected (3). If a mesh function $w(p)$ satisfies the conditions*

$$(\mathcal{L} + c)w(p) \geq 0 \ (\leq 0), \quad p \in \Omega^h, \quad w(p) \geq 0 \ (\leq 0), \quad p \in \partial\Omega^h,$$

then $w(p) \geq 0$ (≤ 0), $p \in \overline{\Omega}^h$.

The proof of the lemma can be found in [6].

3. Monotone iterative methods

Assume that $f(p, v)$ from (1) satisfies the two-sided constraint

$$c_* \leq f_v(p, v) \leq c^*, \quad (p, v) \in \overline{\Omega}^h \times (-\infty, \infty), \quad (f_v \equiv \partial f/\partial v), \tag{5}$$

where c_* and c^* are positive constants.

We say that $\overline{v}(p)$ is an upper solution of (1) if it satisfies the inequalities

$$\mathcal{L}\overline{v}(p) + f(p, \overline{v}) \geq 0, \quad p \in \Omega^h, \quad \overline{v} \geq g \quad \text{on } \partial\Omega^h.$$

Similarly, $\underline{v}(p)$ is called a lower solution if it satisfies all the reversed inequalities. Upper and lower solutions satisfy the inequality

$$\underline{v}(p) \leq \overline{v}(p), \quad p \in \overline{\Omega}^h. \tag{6}$$