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Some Results on the Polar Derivative of a Polynomial

Abdullah Mir* and Bilal Dar

Department of Mathematics, University of Kashmir, Srinagar 190006, India

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Abstract. Let $P(z)$ be a polynomial of degree n and for any complex number α, let $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ denote the polar derivative of $P(z)$ with respect to α. In this paper, we obtain certain inequalities for the polar derivative of a polynomial with restricted zeros. Our results generalize and sharpen some well-known polynomial inequalities.

Key Words: Polynomial, zeros, polar derivative, Bernstein inequality.

AMS Subject Classifications: 30A10, 30C10, 30D15

1 Introduction and statement of results

If $P(z)$ is a polynomial of degree n, then concerning the estimate of $|P'(z)|$ on the unit disk $|z|=1$, we have

$$\max_{|z|=1}|P'(z)| \le n \max_{|z|=1}|P(z)|. \tag{1.1}$$

The above inequality is an immediate consequence of Bernstein's inequality [4] for the derivative of trigonometric polynomial and is best possible with equality holding for all those polynomials having zeros at the origin.

If we restrict ourselves to the class of polynomials having no zeros in $|z|<1$, then the above inequality can be sharpened. In fact, Erdös conjectured and later Lax [11] proved that if $P(z) \neq 0$ in $|z|<1$, then

$$\max_{|z|=1}|P'(z)| \le \frac{n}{2} \max_{|z|=1}|P(z)|. \tag{1.2}$$

*Corresponding author. *Email addresses:* mabdullah_mir@yahoo.co.in (A. Mir), darbilal85@ymail.com (B. Dar)

If the polynomial $P(z)$ of degree n has all its zeros in $|z|\le 1$, then it was proved by Túran [16] that

$$\max_{|z|=1}|P'(z)|\ge\frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1.3}$$

The inequalities (1.2) and (1.3) are best possible and become equality for polynomials which have all its zeros on $|z|=1$.

As an extension of (1.2) and (1.3), Malik [12] proved that if $P(z)\neq 0$ in $|z|<k$, $k\ge 1$, then

$$\max_{|z|=1}|P'(z)|\le\frac{n}{1+k}\max_{|z|=1}|P(z)|, \tag{1.4}$$

whereas if $P(z)$ has all its zeros in $|z|\le k$, $k\le 1$, then

$$\max_{|z|=1}|P'(z)|\ge\frac{n}{1+k}\max_{|z|=1}|P(z)|. \tag{1.5}$$

As a generalization of (1.4), Aziz and Shah [1] proved that if $P(z)$ has no zero in $|z|<k$, $k\ge 1$ except with s- fold zeros at the origin, then

$$\max_{|z|=1}|P'(z)|\le\frac{n+ks}{1+k}\max_{|z|=1}|P(z)|. \tag{1.6}$$

Chan and Malik [5] generalized (1.4) in a different direction and proved that, if

$$P(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu,\quad \mu\ge 1,$$

is a polynomial of degree n which does not vanish in $|z|<k$ where $k\ge 1$, then

$$\max_{|z|=1}|P'(z)|\le\frac{n}{1+k^\mu}\max_{|z|=1}|P(z)|. \tag{1.7}$$

As a refinement of (1.7), Aziz and Shah [3] proved under the same hypothesis that

$$\max_{|z|=1}|P'(z)|\le\frac{n}{1+k^\mu}\Big(\max_{|z|=1}|P(z)|-\min_{|z|=k}|P(z)|\Big). \tag{1.8}$$

On the other hand, for the class of polynomials

$$P(z)=a_nz^n+\sum_{v=\mu}^{n}a_{n-v}z^{n-v},\quad 1\le\mu\le n,$$

of degree n having all its zeros in $|z|\le k$, $k\le 1$, Aziz and Shah [2] proved

$$\max_{|z|=1}|P'(z)|\ge\frac{n}{1+k^\mu}\Big(\max_{|z|=1}|P(z)|+\frac{1}{k^{n-\mu}}\min_{|z|=1}|P(z)|\Big). \tag{1.9}$$