Anal. Theory Appl.
Vol. 27, No. 2 (2011), 150–157
DOI10.1007/s10496-011-0150-3

ON EXTREMAL PROPERTIES FOR THE POLAR DERIVATIVE OF POLYNOMIALS

K. K. Dewan and Arty Ahuja

(*Natural Sciences Jamia Millia Islamia, Central Uniersity, India*)

Received June 23, 2010

Abstract. If $p(z)$ is a polynomial of degree n having all its zeros on $|z|=k, k\le 1$, then it is proved[5] that

$$\max_{|z|=1}|p'(z)|\le \frac{n}{k^{n-1}+k^n}\max_{|z|=1}|p(z)|.$$

In this paper, we generalize the above inequality by extending it to the polar derivative of a polynomial of the type $p(z)=c_nz^n+\sum_{j=\mu}^{n} c_{n-j}z^{n-j}$, $1\le\mu\le n$. We also obtain certain new inequalities concerning the maximum modulus of a polynomial with restricted zeros.

Key words: *polynomial, zeros, inequality, polar derivative*

AMS (2010) subject classification: 30A10, 30C10, 30C15

1 Introduction

If $p(z)$ is a polynomial of degree n and $p'(z)$ its derivative, then according to a famous result known as Bernstein's inequality (for reference see [2]), we have

$$\max_{|z|=1}|p'(z)|\le n\max_{|z|=1}|p(z)|. \tag{1.1}$$

The result is sharp and the equality in (1.1) holds for $p(z)=\lambda z^n$, where $|\lambda|=1$.

For the class of polynomials not vanishing in $|z|<k$, $k\ge 1$, Malik[8] proved

$$\max_{|z|=1}|p'(z)|\le \frac{n}{1+k}\max_{|z|=1}|p(z)|. \tag{1.2}$$

The result is sharp and the extremal polynomial is $p(z)=(z+k)^n$.

While seeking for an inequality analogous to (1.2) for polynomials not vanishing in $|z|<k$, $k\le 1$, Govil[5] proved the following

Theorem A. *If* $p(z)=\sum_{j=0}^{n}c_jz^j$ *is a polynomial of degree n having all its zeros on* $|z|=k$, $k\le 1$, *then*

$$\max_{|z|=1}|p'(z)|\le\frac{n}{k^{n-1}+k^n}\max_{|z|=1}|p(z)|.\tag{1.3}$$

Let α *be a complex number. If* $p(z)$ *is a polynomial of degree n, then the polar derivative of* $p(z)$ *with respect to the point* α, *denoted by* $D_\alpha p(z)$, *is defined by*

$$D_\alpha p(z)=np(z)+(\alpha-z)p'(z).\tag{1.4}$$

Clearly $D_\alpha p(z)$ *is a polynomial of degree at most* $n-1$ *and it generalizes the ordinary derivative in the sense that*

$$\lim_{\alpha\to\infty}\left[\frac{D_\alpha p(z)}{\alpha}\right]=p'(z).\tag{1.5}$$

In this paper, we first prove the following result which is an extension of Theorem A due to Govil[5] to the polar derivative of a polynomial of the type $p(z)=c_nz^n+\sum_{j=\mu}^{n}c_{n-j}z^{n-j}$, $1\le\mu\le n$.

Theorem 1. *If* $p(z)=c_nz^n+\sum_{j=\mu}^{n}c_{n-j}z^{n-j}$, $1\le\mu<n$, *is a polynomial of degree n having all its zeros on* $|z|=k$, $k\le 1$, *then for every real or complex number* α *with* $|\alpha|\ge k$, *we have*

$$\max_{|z|=1}|D_\alpha p(z)|\le\frac{n(|\alpha|+k^\mu)}{k^{n-2\mu+1}+k^{n-\mu+1}}\max_{|z|=1}|p(z)|.\tag{1.6}$$

Instead of proving Theorem 1 we prove the following theorem which gives a better bound over the above theorem. More precisely, we prove.

Theorem 2. *If* $p(z)=c_nz^n+\sum_{j=\mu}^{n}c_{n-j}z^{n-j}$, $1\le\mu<n$, *is a polynomial of degree n having all its zeros on* $|z|=k$, $k\le 1$, *then for every real or complex number* α *with* $|\alpha|\ge k$, *we have*

$$\max_{|z|=1}|D_\alpha p(z)|\le\frac{n(|\alpha|+S_\mu)}{k^{n-2\mu+1}+k^{n-\mu+1}}\max_{|z|=1}|p(z)|,\tag{1.7}$$

where

$$S_\mu=\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{n|c_n|k^{\mu-1}+\mu|c_{n-\mu}|}.\tag{1.8}$$

To prove that the bound obtained in the above theorem is better than the bound obtained in Theorem 1, we show that

$$S_\mu\le k^\mu\quad\text{or}\quad\frac{n|c_n|k^{2\mu}+\mu|c_{n-\mu}|k^{\mu-1}}{\mu|c_{n-\mu}|+n|c_n|k^{\mu-1}}\le k^\mu$$