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SECOND ORDER DECOUPLED IMPLICIT/EXPLICIT METHOD OF THE PRIMITIVE EQUATIONS OF THE OCEAN I: TIME DISCRETIZATION

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Abstract. In this article, we propose the time discretization of the second order decoupled implicit/explicit method of the 3D primitive equations of the ocean in the case of the Dirichlet boundary conditions on the side. We deduce the second order optimal error estimates on the L^2 and H^1 norms of the time discrete velocity and density and the L^2 norm of the time discrete pressure under the restriction of the time step $0 < \tau \leq \beta$ for some positive constant β. Also, we deduce some stability results on the time discrete solution under the same restriction on the time step.

Key words. Primitive equations of the ocean, stability, optimal error estimate, second order decoupled implicit/explicit method.

1. Introduction

Given a smooth bounded domain $\omega \subset R^2$ and the cylindrical domain $\Omega = \omega \times (-d, 0) \subset R^3$, consider in Ω the following 3D viscous primitive equations of the ocean:

$$u_t + L_1 u + (u \cdot \nabla)u + w\partial_z u + \nabla P + f\vec{k} \times u = F_1, \tag{1.1}$$

$$\theta_t + L_2\theta + (u \cdot \nabla)\theta + w\partial_z\theta - \sigma w = F_2 \tag{1.2}$$

$$\nabla \cdot u + \partial_z w = 0, \tag{1.3}$$

$$\partial_z P + \gamma\theta = 0. \tag{1.4}$$

The unknowns for the 3D viscous PEs are the fluid velocity flied $(u, w) = (u_1, u_2, w) \in R^3$ with $u = (u_1, u_2)$ being horizontal, the density θ and the pressure P. Here $f = f_0(\beta + y)$ is the given coriolis rotation frequency with β-plane approximation, F_1 and F_2 are two given functions and $\vec{k}$ is vertical unit vector and $\sigma > 0$ and $\gamma > 0$ are given constant. The elliptic operators L_1 and L_2 are given respectively as the following:

$$L_i = -\nu_i\Delta - \mu_i\partial_z^2, \; i = 1, \; 2.$$

Here the positive constants $\nu_1, \; \mu_1$ are the horizontal and vertical viscosity coefficients; while the positive constants $\nu_2, \; \mu_2$ are the horizontal and vertical thermal diffusivity coefficients and

$$u_t = \frac{\partial u}{\partial t}, \; \theta_t = \frac{\partial \theta}{\partial t}, \; \nabla = (\partial_x, \partial_y), \; \Delta = \partial_{xx} + \partial_{yy}, \; \partial_{x_i} = \frac{\partial}{\partial x_i}, \; \partial_{x_i x_i} = \partial_{x_i}^2,$$

with $i = 1, 2, 3$ and $(x_1, x_2, x_3) = (x, y, z)$.

We partition the boundary of Ω into the following three parts:

$$\Gamma_u = \{(x, y, z) \in \bar{\Omega}; z = 0\}, \; \Gamma_b = \{(x, y, z) \in \bar{\Omega}; z = -d\},$$

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$$\Gamma_s = \{(x,y,z) \in \bar{\Omega}; (x,y) \in \partial\omega,\ -d \le z \le 0\}.$$

We consider the following homogenous boundary conditions of the 3D viscous PEs as in [3, 13, 19]:

$$w|_{\Gamma_u \cup \Gamma_b} = 0. \tag{1.5}$$

$$\partial_z u|_{\Gamma_u \cup \Gamma_b} = 0,\ u \cdot n|_{\Gamma_s} = 0,\ \frac{\partial u}{\partial n} \times n|_{\Gamma_s} = 0; \tag{1.6-1}$$

$$\text{or } u_z|_{\Gamma_u \cup \Gamma_b} = 0,\ u|_{\Gamma_s} = 0; \tag{1.6-2}$$

$$\partial_z \theta|_{\Gamma_b} = (\partial_z \theta + \alpha\theta)|_{\Gamma_u} = 0,\ \frac{\partial \theta}{\partial n}|_{\Gamma_s} = 0. \tag{1.7}$$

Here n is the normal vector of Γ_s, α is a positive constant. Also, the initial conditions of $u(x,y,z,t)$ and $\theta(x,y,z,t)$ should be given by

$$u(x,y,z,0) = u_0(x,y,z),\ \theta(x,y,z,0) = \theta_0(x,y,z). \tag{1.8}$$

Using the Dirichlet boundary condition (1.5) of w on $\Gamma_u \cap \Gamma_b$ and (1.3)-(1.4), we have

$$w(x,y,z,t) = -\int_{-d}^{z} \nabla \cdot u(x,y,\xi,t)d\xi,\ \int_{-d}^{0} \nabla \cdot u(x,y,\xi,t)d\xi = 0,$$

$$P(x,y,z,t) = p(x,y,t) - \gamma \int_{-d}^{z} \theta(x,y,\xi,t)d\xi.$$

With the above statements, one obtains the initial boundary value problem of the 3D viscous PEs:

$$u_t + L_1 u + \nabla p(x,y,t) - \gamma \int_{-d}^{z} \nabla\theta(x,y,\xi,t)d\xi + f\vec{k} \times u$$

$$+ (u \cdot \nabla)u - (\int_{-d}^{z} \nabla \cdot u(x,y,\xi,t)d\xi)\partial_z u = F_1, \tag{1.9}$$

$$\theta_t + L_2\theta + \sigma \int_{-d}^{z} \nabla \cdot u(x,y,\xi,t)d\xi + (u \cdot \nabla)\theta - (\int_{-d}^{z} \nabla \cdot u(x,y,\xi,t)d\xi)\partial_z \theta$$

$$= F_2, \tag{1.10}$$

$$\nabla \cdot \bar{u} = 0, \tag{1.11}$$

together with the boundary condition (1.6)-(1.7) and the initial condition (1.8), where

$$\bar{\phi}(x,y) = \frac{1}{d}\int_{-d}^{0} \phi(x,y,z)dz,\ \tilde{\phi} = \phi - \bar{\phi},$$

for any function $\phi(x,y,z)$ in Ω.

Remark. Recall [3, 13], $F_1 = 0$ and $\gamma = 1$ in (1.9), $\gamma = 1$ in (1.4) and $\sigma = 0$ in (1.2) and (1.10), the boundary condition on u is (1.6-1). While in [19], the boundary condition on u is (1.6-2).

The 3D viscous PEs are very important research subjects in the field of geophysical fluid dynamics, at both the theoretical and numerical levels. There are some well-known difficulties associated with this fundamental equation for 3D oceanic model since their strong nonlinearity.The Mathematical study of the PEs originates in a series of articles, by Lions, Temam and Wang in the early 1990s: see, for instance, [16, 17, 18], where the mathematical formulation of the PEs, which resembles that of the Navier-Stokes equations, was established. Also, the asymptotic analysis and the finite dimensional behavior of the 3D viscous PEs in thin domain as the depth of the domain goes to zero were studied in [11, 12]. For a more extensive discussion and review on this subject, the reader is referred to the