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A TOTAL VARIATION DISCONTINUOUS GALERKIN APPROACH FOR IMAGE RESTORATION

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Abstract. The focal point of this paper is to propose and analyze a $\mathbb{P}_0$ discontinuous Galerkin (DG) formulation for image denoising. The scheme is based on a total variation approach which has been applied successfully in previous papers on image processing. The main idea of the new scheme is to model the restoration process in terms of a discrete energy minimization problem and to derive a corresponding DG variational formulation. Furthermore, we will prove that the method exhibits a unique solution and that a natural maximum principle holds. In addition, a number of examples illustrate the effectiveness of the method.

Key words. Discontinuous Galerkin methods, total variation image denoising, energy minimization methods, nonlinear elliptic filters.

1. Introduction

We consider a two dimensional display on which computer images can be visualized. Mathematically, this can be written in the form $\mathcal{S} = \{K\}$, where $\mathcal{S}$ corresponds to a screen, and $\{K\}$ is a set of disjoint open squares $\{K\}$ representing the pixels. A grayscale image on $\mathcal{S}$ is expressed in terms of a function

$$f : \mathcal{S} \to R,$$

where $R \subset \mathbb{R}$ is a compact interval that is determined by the range of all available grayscale values of the given visualization device.

Graphical data, as evaluated in medical applications, is typically bound to exhibit certain forms of defects which, when visualized on a computer screen, may manifest themselves, for example, in form of *noise* effects. A typical approach is to write

$$f = u + \eta,$$

where f is the perturbed image, u represents the denoised data, and η signifies some noise. A more general approach is given by $f = k * u + \eta$, where k is a convolution kernel that models certain types of blurring. In order to extract the relevant information (such as edges) behind image deficiencies, it is necessary to somehow restore the image f. More precisely, the improved image u is, on the one hand, supposed to contain the essential information of the original image f, and, on the other hand, should be cleaned from disturbing noise or blur.

The problem of image restoration has been tackled by means of several different techniques; let us refer the reader to the overview paper [3] for details. The approach that will be pursued in the present article is based on total variation image denoising. Such models have proved to be particularly effective in edge detection, see, e.g., [2, 5, 6, 9, 10, 11, 12, 13, 14]. Here, as remarked earlier, the basic idea is to take two (competing) aspects into account: First of all, the processed image u is supposed to be close to the original data f. This is accomplished, for instance, by controlling the difference between the two data sets with respect to a certain

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distance measure $\|.\|_{\mathcal{S}}$ on $\mathcal{S}$, i.e., $\|u-f\|_{\mathcal{S}}$. Furthermore, it is desirable that the new image u is suitably denoised when compared to f. In variation based image denoising, supposing for a moment that u is sufficiently smooth on $\mathcal{S}$, this is usually modeled by appropriately bounding the gradient of u, e.g., in the form

$$\int_{\mathcal{S}} \phi(|\nabla u|)\,\mathrm{d}\boldsymbol{x}.$$

Here,

$$\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$$

is a given function that determines the smoothing process. Combining these two issues, an image restoration formulation is now obtained by minimizing the functional

$$\mathsf{F}(u) = \|u-f\|_{\mathcal{S}} + \alpha \int_{\mathcal{S}} \phi(|\nabla u|)\,\mathrm{d}\boldsymbol{x}, \tag{1}$$

where $\alpha \geq 0$ is a fixed constant, with respect to some appropriate function space. Subsequently, by applying standard variational calculus techniques, the above problem can be transformed into an Euler-Lagrange partial differential equation (PDE) formulation which, in turn, can be solved numerically.

In the present article, given the finite dimensional nature of computer images, we propose to formulate the minimization problem (1) directly on a discrete level, i.e., without employing the continuous PDE setting; see also, e.g., [4] for a related approach based on finite difference schemes. Here, we will introduce an energy functional on the discrete space $\mathbb{P}_0$ of all pixel-wise constant functions that features similar properties as the continuous functional F from (1). Then, a corresponding discrete variational formulation is derived. The novelty of our approach is to apply the framework of discontinuous Galerkin methods (see, e.g. [1] and the references therein), which constitute a natural choice in dealing with the discontinuous nature of pixel images, particularly in the context of edge detection. Moreover, the variational framework provides a quite handy setting for the analysis of the well-posedness of the proposed scheme and the derivation of a maximum principle.

The article is outlined as follows: In Section 2 the discrete model is presented. Furthermore, in Section 3 a discontinuous Galerkin formulation is derived and its well-posedness is established. In addition, a maximum principle for the method under consideration will be proved and a number of numerical examples illustrating our approach will be given in Section 4. Finally, we add a few concluding remarks in Section 5.

2. A Discrete Image Denoising Model

In the following section, we will establish a suitable mathematical setting, and the discrete image restoration model will be introduced.

2.1. Mathematical framework. Throughout the manuscript, we will suppose that the screen $\mathcal{S}$ consists of $m \times n$ (open and disjoint) pixels $K_{i,j} \subset \mathbb{R}^2$,

$$\mathcal{S} = \{K_{i,j}\}_{1\leq i\leq m, 1\leq j\leq n},$$

such that

$$\bigcup_{i,j} \overline{K}_{i,j}$$

is a closed rectangle. Each pixel is an open axiparallel square of length $h>0$ that is given by

$$K_{i,j} = ((i-1)h, ih) \times ((j-1)h, jh), \qquad 1 \leq i \leq m,\ 1 \leq j \leq n.$$