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A SPECTRAL METHOD FOR MIXED BOUNDARY VALUE PROBLEMS ON HEXAHEDRONS

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Abstract. In this paper, we investigate a spectral method for mixed boundary value problems defined on hexahedrons. Some results on irrational orthogonal approximation are established, which play important roles in numerical solutions of partial differential equations defined on hexahedrons. As examples of applications, we provide spectral schemes for two model problems, and prove their spectral accuracy. Efficient numerical implementations are described. Numerical results demonstrate the high efficiency of suggested algorithms.

Key words. Irrational orthogonal approximation, spectral method on hexahedrons, mixed boundary value problems.

1. Introduction

Over the past three decades, spectral methods have been increasingly popular in scientific computations. Especially, the Legendre and Chebyshev spectral methods have been widely used for numerical solutions of partial differential equations, see [1, 2, 3, 7, 8, 11, 13, 18] and the references therein. Recently, there was also some work on the Jacobi approximation and the Jacobi spectral method for degenerated problems, see [9, 10, 14, 15, 16]. Most of the problems considered in these papers are defined on bounded rectangular domains. However, it is more practical to deal with problems defined on non-rectangular domains. In particular, it is interesting to develop the spectral method for three-dimensional and non-rectangular domains. Recently, Guo and Jia [12] proposed a spectral method and a spectral element method on polygonal domains. Whereas, so far, there has been little work on spectral and spectral element methods for hexahedrons and polyhedrons.

In this paper, we investigate the spectral method for mixed boundary value problems on hexahedrons. We first recall some recent results on the Legendre orthogonal approximation on the cube in the next section. Then, we introduce the irrational orthogonal approximation on arbitrary convex hexahedrons and establish the basic results on such approximation in Section 3. These results play essential roles in numerical solutions of partial differential equations defined on hexahedrons. As applications of the above results, we propose the spectral method for two model problems on hexahedrons in Section 4. Their spectral accuracy is proved. We describe the numerical implementation of proposed schemes in Section 5, together with some numerical results to demonstrate the high efficiency of our algorithms. The last section is for some concluding remarks. The main idea and techniques

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developed in this work are also applicable to other mixed boundary value problems defined on three-dimensional and non-rectangular domains.

2. Preliminaries

In this section, we recall some recent results on the Legendre orthogonal approximation in three-dimensions. Let the interval $I_j = \{ \xi_j \mid -1 < \xi_j < 1\}$ and the cube $K = \{ \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \mid \xi_j \in I_j,\ 1 \le j \le 3\}$. We denote by $H^r(K)$ and $H_0^r(K)$ the Sobolev spaces as usual with the norm $||u||_{r,K}$. The inner product and the norm of $L^2(K)$ are denoted by $(u, v)_K$ and $||u||_K$, respectively.

For any integer $N > 0, \mathcal{P}_N(I_1)$ stands for the set of all polynomials of degree at most N, and

$$V_N(K) = \mathcal{P}_N(I_1) \otimes \mathcal{P}_N(I_2) \otimes \mathcal{P}_N(I_3).$$

The $L^2(K)-$orthogonal projection $P_{N,K} : L^2(K) \to V_N(K)$ is defined by

$$(u - P_{N,K}u, \phi)_K = 0, \qquad \forall \phi \in V_N(K).$$

For describing the approximation error precisely, we introduce the following quantity with an integer $r \ge 0$,

$$\begin{aligned}\mathbb{A}_{r,K}(u) &= \int_{I_3}\int_{I_2} \|(1-\xi_1^2)^{\frac{r}{2}}\partial_{\xi_1}^r u(\cdot,\xi_2,\xi_3)\|_{I_1}^2 d\xi_2 d\xi_3 \\ &+ \int_{I_3}\int_{I_1} \|(1-\xi_2^2)^{\frac{r}{2}}\partial_{\xi_2}^r u(\xi_1,\cdot,\xi_3)\|_{I_2}^2 d\xi_1 d\xi_3 \\ &+ \int_{I_2}\int_{I_1} \|(1-\xi_3^2)^{\frac{r}{2}}\partial_{\xi_3}^r u(\xi_1,\xi_2,\cdot)\|_{I_3}^2 d\xi_1 d\xi_2.\end{aligned}$$

Throughout this paper, we denote by c a generic constant independent of any function and N. According to Theorem 2.1 of [19], we know that if $u \in L^2(K)$, and $\mathbb{A}_{r,K}(u)$ is finite for integers $r \ge 0, r \le N+1$, then

$$\|P_{N,K}u - u\|_K^2 \le cN^{-2r}\mathbb{A}_{r,K}(u). \tag{1}$$

Next, let $V_N^0(K) = H_0^1(K) \cap V_N(K)$. The $H_0^1(K)-$orthogonal projection $P_{N,K}^{1,0}$: $H_0^1(K) \to V_N^0(K)$ is defined by

$$(\nabla(P_{N,K}^{1,0}u - u), \nabla\phi)_K = 0, \qquad \forall \phi \in V_N^0(K).$$

For any integer $r \ge 1$, we define

$$\mathbb{B}_{r,K}(u) = \mathbb{B}_{r,K}^{(1)}(u) + \mathbb{B}_{r,K}^{(2)}(u) + \mathbb{B}_{r,K}^{(3)}(u), \tag{2}$$

where for $r = 1, 2$,

$$\mathbb{B}_{r,K}^{(1)}(u) = \mathbb{B}_{r,K}^{(2)}(u) = \mathbb{B}_{r,K}^{(3)}(u) = ||u||_{r,K}^2,$$

and for $r \ge 3$,

(3)

$$\mathbb{B}_{r,K}^{(1)}(u) = \int\int\int_K ((1-\xi_1^2)^{r-1}(\partial_{\xi_1}^r u)^2 + (1-\xi_2^2)^{r-1}(\partial_{\xi_2}^r u)^2 + (1-\xi_3^2)^{r-1}(\partial_{\xi_3}^r u)^2) d\xi_1 d\xi_2 d\xi_3,$$

$$\begin{aligned}\mathbb{B}_{r,K}^{(2)}(u) &= \int\int\int_K (1-\xi_1^2)^{r-2}((\partial_{\xi_1}^{r-1}\partial_{\xi_2}u)^2 + (\partial_{\xi_1}^{r-1}\partial_{\xi_3}u)^2) d\xi_1 d\xi_2 d\xi_3 \\ &+ \int\int\int_K (1-\xi_2^2)^{r-2}((\partial_{\xi_1}\partial_{\xi_2}^{r-1}u)^2 + (\partial_{\xi_2}^{r-1}\partial_{\xi_3}u)^2) d\xi_1 d\xi_2 d\xi_3 \\ &+ \int\int\int_K (1-\xi_3^2)^{r-2}((\partial_{\xi_2}\partial_{\xi_3}^{r-1}u)^2 + (\partial_{\xi_1}\partial_{\xi_3}^{r-1}u)^2) d\xi_1 d\xi_2 d\xi_3,\end{aligned} \tag{4}$$