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On the Lorenz Systems for the Incompressible Flow between Two Concentric Rotating Cylinders

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Abstract. A spectral method is used to derive a series of equations for axisymmetric Couette-Taylor flow. A three-modes system, which is similar to the Lorenz systems, is obtained by a suitable three-modes truncation of the Navier-Stokes equations for the incompressible flow between two concentric rotating cylinders. The stability of the three-modes systems is discussed. Moreover, the existence of its attractor and the estimation of Hausdorff dimension are given.

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1 Introduction

The Navier-Stokes (NS) equations have been used for describing fluid flows. It is a very difficult issue to understand the stability of the Navier-Stokes equations. For example, the stability of flow between two concentric cylinders (abbreviate frequently as Couette-Taylor flow) is an open problem. There have been many investigations concerning with the rotating Couette-Taylor flow, see, e.g., [1–9]. It is a fundamental problem of fluid mechanics, and has extensive applications in lubricating theory and engineering. As the Reynolds number is increased, this flow loses its stability, and displays some behaviors of nonlinear dynamics. Accordingly, it is very important to study the Couette-Taylor flow problem. The Lorenz truncation method is used to derive a model system of evolutionary equations for the axisymmetric Couette-Taylor flow in [1]. The advantage of the

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method is to use the simple model to describe the features of the complicate problems, but the model seems too simple. In this paper, we derive a model system by using spectral methods. First, we introduce the analytical expressions of the eigenfunction of the Stokes operator in the cylindrical gap region, and derive Lorenz systems containing only three Fourier modes by using the eigenfunction of the Stokes operator. More precisely, we first obtain a model system by taking three modes in the Fourier expansion, which is similar to the Lorenz systems. Secondly, we discuss the stability for stationary solution of the three-modes systems, the existence of its attractor and the estimation of the Hausdorff dimension. By using the model truncation method, we can not only reduce the complexity of the spectral method, but also find very interesting phenomena of the model systems. It is of enlightenment to discuss nonlinear phenomena of bifurcation and turbulence for the Navier-Stokes equations. We point out that the method of expressing some features of complicated problems by using simple model system is very valuable. By introducing eigenfunctions of Stokes operator as the basis of finite dimensional approximate subspaces, our spectral method is more appropriate for the Couette-Taylor flow problem. In fact, if the eigenfunctions of the Stokes operator is regarded as basic flow (basic vortices), then the Couette-Taylor flow is accumulation of various basic vortices. The method in [1] uses different ways for different gaps between two concentric cylinders, whereas our method uses a uniform way for different cylinder gaps so that our discussion and result are concise.

The content of the paper is arranged as follows. We first introduce eigenvalue and eigenfunctions of the Stokes operator in the cylindrical gap regions in Section 2. In Section 3, we will derive a three-modes Lorenz system by a suitable mode truncation of the Navier-Stokes equations for the incompressible flow between two concentric rotating cylinders, and then discuss the stability of the three-modes system. In Section 4, we will present the existence of the attractors for the system and estimate the relevant Hausdorff dimension.

2 Stream-function for the NS equations and spectrum of the Stokes operator

We consider the incompressible flow between two concentric rotating cylinders. We first introduce some notations: (r,φ,z) is the cylindrical coordinate; r_1, r_2 are radius of the inner and outer cylinder, respectively; Ω_1,Ω_2 are angular velocity of the inner and outer cylinder, respectively; $\mu=\Omega_2/\Omega_1$ is the angular velocity ratio of the inner and outer cylinder; $\eta=r_1/r_2$ is the radius ratio of the inner and outer cylinder; ν is the coefficient of kinematic viscosity; $R=\Omega_1 r_1^2/\nu$ is the Reynolds number; ψ is the stream function; $(u_r,u_\varphi,u_z),p$ are velocity of the fluids in cylindrical coordinate and pressure, respectively.

Under assumption of periodic boundary condition in the Z-axis, we introduce fol-