J. Partial Diff. Eqs. **19**(2006), 304–318
©International Academic Publishers Vol.19 No.4

A NOTE ON L^2 DECAY OF LADYZHENSKAYA MODEL

Dong Boqing
(School of Mathematical Sciences, Nankai University, Tianjin 300071, China)
(E-mail: bqdong@mail.nankai.edu.cn)
(Received Mar. 28, 2005; revised Jul. 17, 2005)

Abstract This paper is concerned with time decay problem of Ladyzhenskaya model governed incompressible viscous fluid motion with the dissipative potential having p-growth ($p \geq 3$) in $\mathbf{R}^3$. With the aid of the spectral decomposition of the Stokes operator and $L^p - L^q$ estimates, it is rigorously proved that the Leray-Hopf type weak solutions decay in $\mathrm{L}^2(\mathbf{R}^3)$ norm like $t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}$ under the initial data $u_0 \in L^2(\mathbf{R}^3) \cap L^r(\mathbf{R}^3)$ for $1 \leq r < 2$. Moreover, the explicit error estimates of the difference between Ladyzhenskaya model and Navier-Stokes flow are also investigated.
Key Words Ladyzhenskaya model; L^2 decay; spectral decomposition.
2000 MR Subject Classification 35B40, 35Q35, 76A05.
Chinese Library Classification O175.29.

1. Introduction

Consider the viscous incompressible fluid motion governed by the following momentum and continuity equations

$$\partial_t u + (u \cdot \nabla)u - \nabla \cdot \tau^v + \nabla \pi = 0 \qquad \text{in } \mathbf{R}^3 \times (0, \infty), \tag{1.1}$$

$$\nabla \cdot u = 0 \qquad \text{in } \mathbf{R}^3 \times (0, \infty) \tag{1.2}$$

together with the boundary and initial conditions

$$\lim_{|x| \to \infty} u(x,t) = 0 \qquad \text{in } (0, \infty), \tag{1.3}$$

$$u(x,0) = u_0 \qquad \text{in } \mathbf{R}^3. \tag{1.4}$$

Here , the gradient $\nabla = (\partial_{x_1}, ..., \partial_{x_3})$, $u = (u_1, ..., u_3)$ and π denote the unknown velocity and pressure of the fluid motion at the point $(x,t) \in \mathbf{R}^3 \times (0, \infty)$, respectively, while u_0 is the given initial velocity vector field. For simplicity, we assume that the external force has a scalar potential and it is included into the pressure gradient. $\tau^v = (\tau^v_{ij})$ is the stress tensor specified in the following form

$$\tau^v_{ij} = 2\ (\mu_0 + \mu_1 |e(u)|^{p-2})\ e_{ij}(u) \tag{1.5}$$

for the symmetric deformation velocity tensor $e(u) = (e_{ij}(u))$ with

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad |e(u)| = (e_{ij}(u)e_{ij}(u))^{\frac{1}{2}} \tag{1.6}$$

where the viscosities $\mu_0 > 0$ and $\mu_1 \geq 0$.

When $\mu_1 = 0$, the Stokes Law

$$\tau_{ij}^v = 2\mu_0 e_{ij}(u) \tag{1.7}$$

holds true. The fluids, such as water and alcohol, satisfying the linear equation expressed by (1.7) are said to be Newtonian, and (1.1) turns out to be the Navier-Stokes equations (refer to [1] for details), whereas the nonlinear constitutive equation expressed by (1.5) with $\mu_1 > 0$ is related to other non-Newtonian fluids such as the molten plastics, dyes, adhesives, paints and greases. Equations (1.1)-(1.6) with $\mu_1 > 0$ were first proposed by Ladyzhenskaya [2] and have been known as the Ladyzhenskaya model which may be justified through a variety of physical and mathematical arguments. Additionally, the constitutive equation expressed by (1.5) is defined by the physical qualities of a fluid and is also called Ellis fluids model when $p > 2$ (refer to Chapter 2 of [3]).

There is extensive literature on the large time behavior of the viscous incompressible fluid flows. On the one hand, as for the Navier-Stokes equations, the decay problem of weak solutions was first proposed by Leray [4]. Schonbek [5] and Wiegner [6] introduced Fourier splitting methods and obtained time decay rates with respect to the whole spaces $\mathbf{R}^n$. Kajikiya and Miyakawa [7] provided a spectral decomposition approach of the Stokes operator and also derived time decay rates in $\mathbf{R}^n$. One may also refer to the study of He *et al* [8, 9] relating to the decay properties for strong solutions of Navier-Stokes equations.

On the other hand, for Ladyzhenskaya model governed incompressible viscous non-Newtonian fluid motions, the existence of weak solutions was obtained by Ladyzhenskaya [2] and J. L. Lions [10] for $p \geq \frac{11}{5}$, and more recently, Du and Gunzburger [11] have studied the somewhat more general existence and uniqueness results in a bounded domains. Pokorny [12] investigated the Cauchy problem for this model in whole spaces. we also refer to the work of [13-15] to the nonlinear multipolar viscous fluids. Additionally, with the aid of Fourier splitting method [5], the time decay problem of Ladyzhenskaya model was recently examined by Necǎsová and Penel [16] for logarithmic decay in $\mathbf{R}^2$ and algebraic decay in $\mathbf{R}^3$. Guo and Zhu [17] improved the algebraic decay results in $\mathbf{R}^n (n \geq 2)$ by the modification of Fourier splitting method [6], more precisely, when $u_0 \in L^2(\mathbf{R}^n) \cap L^r(\mathbf{R}^n)$ for $1 \leq r < 2$, they have obtained the weak solutions decay as follows

$$\|u(t)\|_{L^2} \leq c(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}, \quad \|u(t) - e^{t\Delta}u_0\|_{L^2} \to 0, \quad t \to \infty. \tag{1.8}$$