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A PROOF OF THE UNIFORMIZATION THEOREM ON S^2*

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Abstract We present a simple proof for the uniformization theorem on 2-sphere by methods of elliptic partial differential equations.
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Let S^2, R^2 and D^2 denote the standard 2-sphere, 2-plane, and unit 2-disk, respectively. The uniformization theorem states that

Any closed Rieman surface M with Euler characteristics χ is conformally covered by either S^2, R^2 or D^2, according to $\chi > 0, = 0$, or < 0.

For the case $\chi > 0, M$ must be diffeomorphic to S^2, and we can translate (through the one-to-one correspondance between the conformal structures and the conformal classes of Riemannian metrics) the above statement to

Theorem A *Given any Riemannian metric g on S^2, there exists a conformal diffeomorphism $f : (S^2, g_0) \to (S^2, g)$, where g_0 denotes the standard Riemannian metric with constant curvature 1, and the conformality of f means $f^*g = e^u g_0$ for some smooth function u on S^2.*

A proof of Theorem A can be found in [1], Chap.3, Theorem 3.1.1, where the author applies a variational approach to the problem.

It is also well known that Theorem A is equivalent to the following (cf.[2], pp.6–7)

Theorem B *Given any Riemannian metric g on S^2, there exists a conformal metric $g' = e^u g$ with constant Gaussian curvature 1.*

Indeed, the equivalence of Theorems A and B is implied by the following well-known result in Riemannian geometry, which we may call "Geometric Uniformization".

Theorem C *Let M be an m-dimensional, simply connnected, complete, Riemannian manifold with constant sectional curvature $K = 1, 0$, or -1. Then M is isometric to the standard sphere S^m, the Euclidean space R^m, or the hyperbolic space H^m.*

Now, assuming Theorem A is true, it is straightforward to see that $(f^{-1})^* g_0$ is a conformal metric claimed in Theorem B. So Theorem B holds true. On the other

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hand, if Theorem B is true, then by Theorem $C, (S^2, e^u g)$ is isometric to (S^2, g_0), i.e. there exists a diffeomorphism h such that $h^* g_0 = e^u g$. It is clear then that $f = h^{-1}$ is the conformal diffeomorphism claimed in Theorem A. Thus Theorems A and B are equivalent.

It is known that to prove Theorem B, one needs only to find a solution to the following elliptic equation.

$$\Delta u - 2k_g + 2e^u = 0 \tag{1}$$

where Δ is the Laplace operator with respect to the metric g and k_g is the Gaussian curvature of g. Actually, the Gaussian curvature k of $e^u g$ is given by (see [3])

$$k = e^{-u}\left(k_g - \frac{1}{2}\Delta u\right) \tag{2}$$

The existence of a solution of (1) was proved by R.Hamilton ([4]) and B. Chow ([5]). They used the so-called Ricci flow, which, in dimension two, is essentially a nonlinear parabolic equation for a single function. They proved that the solutions of the Ricci flow exist for all time and converges as time goes to infinity to the solutions of (1).

It is remarkable that for the equation (1) on S^2, simple as it is, there has not been an existence proof by simple elliptic methods. Even after the Yamabe problem, which can be viewed as the higher dimensional extension of Theorem B, was completely resolved by R.Schoen ([6]), a proof comparable to Schoen's for Theorem B is still lacking. It is the aim of this note to present such a proof. In fact, the follwing proof was discovered by this author early in 1988, and was supposed to publish in a book. Unfortunately the publication of that book was cancelled later.

Proof of Theorem B Without loss of generality we may assume that the total area of the metric g is 4π. Choose any point $P \in S^2$, we first solve the equation

$$\Delta\varphi - 2k_g = 0 \quad \text{on } S^2 \backslash \{P\} \tag{3}$$

Let G be the Green's function ([3], Chap.4) on S^2 with pole at P. Then G satisfies the equation

$$\Delta G - 1 = 4\pi\delta_P \tag{4}$$

where δ_P is the delta measure at P. Next, let v be a smooth function that solves

$$\Delta v = 2k_g - 2$$

The existence of such a v is because the mean value of the right hand side of the above equation equals zero, a fact guaranteed by the Gauss-Bonnet formula

$$\int_{S^2} k_g dA_g = 4\pi \tag{5}$$

Now, letting $\varphi = 2G + v$ we see φ solves the equation (3). By (2), the metric $g_1 = e^{\varphi} g$ has constant curvature zero. Moreover, the asymptotic behaviour of G at P (see [3])