J.Partial Diff. Eqs. **10**(1997), 333–346
©International Academic Publishers Vol.10, No.4

ENTROPY METHODS FOR NONSTRICTLY HYPERBOLIC SYSTEMS

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Dedicated to Professor Ding Xiaxi on the Occasion of His 70th Birthday

(Received June 6, 1997)

Abstract This paper regards the existence of entropy waves for a model example of nonstrictly hyperbolic system which degenerates on a sonic line. This construction is an important tool to approach the convergence of the vanishing viscosity and of other similar approximations to the weak solutions of the hyperbolic systems, via the compensated compactness techniques.

Key Words Conservation laws; entropy; Goursat problem; nonlinear wave equation; Riemann function; sonic line; weak and strong fundamental solutions.

Classification 35L65, 35L05.

1. Introduction

This paper is concerned with the study of entropy waves for the nonstrictly hyperbolic quasilinear wave equation. This is a model problem for a mathematical difficulty in the general theory of nonstrictly hyperbolic systems which has not completely understood.

This particular case embodies many of these difficulties and can be considered a good prototype for the theory.

In particular we are interested to hyperbolic systems which exhibit a degeneracy of strict hyperbolicity along a sonic line. The investigation of the entropy waves is an essential tool in order to apply the techniques of compensated compactness (see [1–9]).

1.1 Generalities

Let us consider the following system of conservation laws

$$\mathcal{U}_t + f(\mathcal{U})_x = 0 \tag{1.1}$$

with Cauchy data

$$\mathcal{U}(x,0) = \mathcal{U}_0(x)$$

where $\mathcal{U} = \mathcal{U}(x,t) \in \mathbf{R}^n$, $(x,t) \in \mathbf{R} \times \mathbf{R}^+$ and f is a smooth nonlinear function defined on an open region $\Omega \subset \mathbf{R}^n$. A weak solution to (1.1) is a bounded measurable function $\mathcal{U} = \mathcal{U}(x,t)$ such that for all $\Phi \in C_0^1$ one has

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} (\mathcal{U}\Phi_t + f(\mathcal{U})\Phi_x)(x,t)dxdt + \int_{-\infty}^{+\infty} \mathcal{U}_0(x)\Phi(x,0)dx = 0$$

The pair of real valued functions (η, q) is called an entropy-entropy flux pair associated to the system (1.1) if $\eta, q \in C^1(\Omega)$ and

$$\nabla\eta\nabla f = \nabla q \tag{1.2}$$

When $\eta \in C^2(\Omega)$ and Ω is simply connected, then (1.2) is equivalent to study

$$\operatorname{curl}(\nabla\eta\nabla f) = 0 \tag{1.3}$$

Let us denote by $\lambda_i(\mathcal{U})$, $i = 1, \cdots, n$ the characteristic speeds of the system (1.1). We say that (1.1) is strictly hyperbolic in Ω if and only if for any i $\lambda_i(\mathcal{U})$ is real and $\lambda_j(\mathcal{U}) \neq \lambda_k(\mathcal{U})$, for $j \neq k$, for all $\mathcal{U} \in \Omega$. If there exists a point $\overline{\mathcal{U}}$ and two indices j, k such that $\lambda_j(\overline{\mathcal{U}}) = \lambda_k(\overline{\mathcal{U}})$, the system is nonstrictly hyperbolic in $\overline{\mathcal{U}}$ which is called an umbilical point. When (1.1) is a nonstrictly hyperbolic system, also the linear equation (1.2) becomes a nonstrictly hyperbolic system of n equations in two unknowns. Finally, the system (1.1) is called genuinely nonlinear if $\nabla\lambda_k \cdot r_k \neq 0$, for all $\mathcal{U}, k = 1, \cdots, n$.

Multiplying the equation (1.2) by the right eigenvectors of the Jacobian matrix Df, we obtain the characteristic form

$$r_j \cdot (\lambda_j \nabla\eta - \nabla q) = 0, \quad j = 1, \cdots, n \tag{1.4}$$

In the special case $n = 2$, by introducing the Riemann invariants coordinate system (ω, z), (1.4) transforms into

$$\begin{cases} \lambda_1 \eta_z = q_z \\ \lambda_2 \eta_\omega = q_\omega \end{cases} \tag{1.5}$$

Also in the characteristic variables, after cross differentiation, we obtain a second order partial differential equation for η:

$$(\lambda_1 - \lambda_2)\eta_{\omega z} + \lambda_{1\omega}\eta_z - \lambda_{2\omega}\eta_\omega = 0 \tag{1.6}$$

Notice that, from (1.6), in any umbilical point for the problem (1.1), there is a singularity since the coefficient of the higher order term vanishes.