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# A COUNTEREXAMPLE IN THE THEORY OF COERCIVENESS FOR ELLIPTIC SYSTEMS

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## Abstract

In this note we exhibit an counterexample to solve an open problem presented by M. Giaquinta negatively. The problem is that if linear second order strongly elliptic systems in the sense of Legendre -Hadamard satisfy weak coerciveness condition, i. e., Gårding's inequality, when the coefficients of the system are in $L^\infty$.

In [3], Giaquinta mentioned the following open problem:

Suppose

1) $A_{ij}^{\alpha\beta}$, $b_{ij}^{\alpha}$, $c_{ij}^{\beta}$, $d_{ij} \in L^\infty(\Omega)$

2) $A_{ij}^{\alpha\beta}(x)\xi_\alpha\xi_\beta\eta^i\eta^j \geq c|\xi|^2|\eta|^2$ for $\eta \in R^N$, $\xi \in R^n$

i. e., Legendre-Hadamard condition, where $c > 0$ is a constant, $\alpha, \beta = 1, \ldots, n$; $i, j = 1, \ldots, N$ $(N > 1)$ and the summation convention is understood.

Define

$$a(u, v) = \int_\Omega (A_{ij}^{\alpha\beta}(x) D_\alpha u^i D_\beta v^j + b_{ij}^\alpha D_\alpha u^i v^j + c_{ij}^\beta u^i D_\beta v^j + $$
$$+ d_{ij} u^i v^j)\, dx, \qquad u, v \in C_0^\infty(\Omega; R^N) \qquad (*)$$

The problem is, under the above assumptions, if $a(\cdot, \cdot)$ is weak coercive, i. e., if there exist $\lambda_0 > 0$ and $\lambda_1$ such that

$$a(u, u) \geq \lambda_0 \int_\Omega |Du|^2 dx - \lambda_1 \int_\Omega |u|^2 dx, \qquad u \in C_0^\infty(\Omega; R^N)$$

This type of problems is the content of Gårding's inequality which is very important in the theory of partial differential equations. It is known that the answer of the above problem is positive when $A_{ij}^{\alpha\beta}(x)$ is uniformly continuous on $\bar{\Omega}$ (see [3, page 12]). For more information on the theory of coerciveness we refer to [1], [4].

In this note we will show by a counterexample, that the answer of the above problem is generally negative when $A_{ij}^{\alpha\beta} \in L^\infty$.

**Notations** We will use notations and conventions of [2]. In particular we adopt the summation convention with $\alpha, \beta$ running from 1 to $n$ and $i, j$ running from 1 to $N$.

**Example** For $n, N \geq 2$, define $B(x, t) = \{y \in R^n, |y - x| < t\}$ for $x \in R^n, t > 0$, and set

$$D_k = B(p_k, 1/2^k), \quad B_k = B(p_k, 1/2^{k+1}), \quad D = B(0, 4)$$

where

$$p_k = (s_k, 0, \dots, 0); \quad s_k = 3(1 - 1/2^k), \ k = 0, 1, 2, \dots$$

Choose $\zeta \in C_0(R^n)$ to satisfy

$$\begin{cases} \zeta = 1 \ \text{ on } \ B(0, 1/2), & \zeta = 0 \ \text{ on } R^n \backslash B(0, 1) \\ 0 \leq \zeta \leq 1, & |D\zeta| \leq C, \ C \ \text{ is a positive constant} \end{cases}$$

Define

$$A_{ij}^{\alpha\beta}(x) = \begin{cases} \delta^{\alpha\beta}\delta_{ij} - (K+2)\,\delta_{ij}^{\alpha\beta} & x \in \bigcup_{k=0}^{\infty} B_k, \ \max\{\alpha, \beta, i, j\} \leq 2 \\ \delta^{\alpha\beta}\delta_{ij} & x \in \bigcup_{k=0}^{\infty} B_k, \ \max\{\alpha, \beta, i, j\} > 2 \\ \delta^{\alpha\beta}\delta_{ij} & x \in D \backslash \bigcup_{k=0}^{\infty} B_k \end{cases}$$

where $\delta^{\alpha\beta}$, $\delta_{ij}$ are Kronecker symbols and

$$\delta_{ij}^{\alpha\beta} = \begin{cases} 1 & \text{if } i \neq j \text{ and } (\alpha, \beta) \text{ is an even permutation of } (i, j) \\ -1 & \text{if } i \neq j \text{ and } (\alpha, \beta) \text{ is an odd permutation of } (i, j) \\ 0 & \text{if } i = j \text{ or } (\alpha, \beta) \text{ is not a permutation of } (i, j) \end{cases}$$

and we will choose $K > 0$ at the end of the proof bellow.

It is obvious that $A_{ij}^{\alpha\beta}(x) \in L^\infty(D)$, such that

$$A_{ij}^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta\eta^i\eta^j = |\xi|^2|\eta|^2, \ \text{for} \ \ \xi \in R^n, \ \eta \in R^N \ \text{and} \ \ x \in D$$

We now prove that for every $\lambda > 0$, there exists $u \in C_0^\infty(D; R^N)$ such that

$$\int_D (A_{ij}^{\alpha\beta}(x)\,D_\alpha u^i D_\beta u^j + \lambda|u|^2)\,dx < 0$$

For the given $\lambda > 0$, choose an integer $m > 0$, such that $2^m > \lambda$. Define

$$v^1(x) = (\exp 2^m(x_1 - s_m))\cos 2^m x_2, \quad v^2(x) = (\exp 2^m(x_1 - s_m))\sin 2^m x_2$$

for $x = (x_1, x_2, \dots, x_n) \in D_m$. $(v^1, v^2)$ is the solution of the Cauchy-Riemann equations (see [5]).

$$v_{x_1}^1 = v_{x_2}^2, \quad v_{x_1}^2 = -v_{x_2}^1 \tag{1}$$

and

$$|Dv| = 2^m|v| \tag{2}$$

Set

$$u^i(x) = \begin{cases} v^i(x)\,\zeta_m(x) & x \in D, \ i = 1, 2 \\ 0 & x \in D, \ 2 < i \leq N \end{cases}$$

where