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High-Order Accurate Runge-Kutta (Local) Discontinuous Galerkin Methods for One- and Two-Dimensional Fractional Diffusion Equations

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Abstract. As the generalization of the integer order partial differential equations (PDE), the fractional order PDEs are drawing more and more attention for their applications in fluid flow, finance and other areas. This paper presents high-order accurate Runge-Kutta local discontinuous Galerkin (DG) methods for one- and two-dimensional fractional diffusion equations containing derivatives of fractional order in space. The Caputo derivative is chosen as the representation of spatial derivative, because it may represent the fractional derivative by an integral operator. Some numerical examples show that the convergence orders of the proposed local P^k–DG methods are $O(h^{k+1})$ both in one and two dimensions, where P^k denotes the space of the real-valued polynomials with degree at most k.

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Key words: Discontinuous Galerkin method, Runge-Kutta time discretization, fractional derivative, Caputo derivative, diffusion equation.

1. Introduction

Fractional calculus is a natural extension of the integer order calculus [28, 30]. Recently many problems in physics [2], finance [31] and hydrology [1] have been formulated on fractional partial differential equations (PDE), containing derivatives of fractional order in space, time or both. For example, anomalous diffusion is a possible mechanism underlying plasma transport in magnetically confined plasmas, and the fractional order space derivative operators can be used to model such transport mechanism.

In recent years the numerical solutions of the fractional PDEs have attracted a considerable interest both in mathematics and in applications. An intrinsical difference between

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the behaviors of integer and fractional order derivatives is that the integer order derivatives depend only on the local behavior of a function or solution, while the fractional derivatives are non-local, i.e., they depend on the entire function or solution. Thus, new difficulties and challenges appear in deriving numerical methods for this kind of equations.

The fractional derivatives of order $\alpha > 0$, $\frac{\partial^\alpha u}{\partial x^\alpha}$, are usually represented by the Riemann-Liouville formula [28,30]

$$\frac{\partial^\alpha u}{\partial x^\alpha}(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_a^x u(\xi,t)(x-\xi)^{n-\alpha-1}\, d\xi, \tag{1.1}$$

where $\Gamma(\cdot)$ is the Gamma function, $x \in [a,b]$, $-\infty \le a < b \le \infty$, $n-1 < \alpha < n$, $n \in \mathbb{Z}^+$. The fractional derivatives are also frequently defined by the Grünwald-Letnikov formula

$$\frac{\partial^\alpha u}{\partial x^\alpha}(x,t) = \lim_{\Delta x \to 0} \frac{1}{\Delta x^\alpha} \sum_{\ell=0}^{[\frac{x-a}{\Delta x}]} (-1)^\ell \begin{pmatrix} \alpha \\ \ell \end{pmatrix} u(x-\ell\Delta x, t), \tag{1.2}$$

where $[\frac{x-a}{\Delta x}]$ denotes the integer part of $\frac{x-a}{\Delta x}$. If $u(\xi,\cdot) \in C^n[a,x]$, the Riemann-Liouville formula is equivalent to the Grünwald-Letnikov. However, the discrete approximations of the latter present some limitations: frequently numerical approximations based on this formula originate unstable numerical methods and henceforth in many cases a shifted Grünwald-Letnikov formula is used; the order of accuracy of such approaches is never higher than one.

Another way to represent the fractional derivative is by the Caputo formula

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{\partial^n u(\xi,t)}{\partial \xi^n}(x-\xi)^{n-\alpha-1} d\xi. \tag{1.3}$$

This formula has some advantages over the Riemann-Liouville formula. The Laplace transform method is very frequently used for solving fractional differential equations, the Laplace transform of the Riemann-Liouville derivatives leads to boundary conditions involving the limit value of the Riemann-Liouville derivatives at the lower terminal $x = a$. Although technically such problems can be solved, there is no physical interpretation. On the other hand, the Laplace transform of the Caputo derivative imposes boundary conditions involving integer order derivatives which usually are more acceptable and physical. Another advantage is that the Caputo derivative of a constant is zero, while for the Riemann-Liouville it's not.

During the past decade, numerical methods of the fractional PDEs have been increasingly appearing in literatures. Lynch et al. [24] studied the numerical properties of the PDEs of fractional order $\alpha \in (1,2)$. Shen and Liu [35] gave error analysis of an explicit finite difference approximation for the space fractional diffusion equation with insulated ends. Chen et al. [3] proved the stability and convergence of an implicit difference approximation scheme of the fractional diffusion equation describing anomalous slow diffusion (sub-diffusion) by using a Fourier method. Liu et al. [21] discussed stability and convergence of the difference methods for the space-time fractional advection-diffusion equation.