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Superconvergence and L^∞-Error Estimates of RT1 Mixed Methods for Semilinear Elliptic Control Problems with an Integral Constraint

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Abstract. In this paper, we investigate the superconvergence property and the L^∞-error estimates of mixed finite element methods for a semilinear elliptic control problem with an integral constraint. The state and co-state are approximated by the order one Raviart-Thomas mixed finite element space and the control variable is approximated by piecewise constant functions or piecewise linear functions. We derive some superconvergence results for the control variable and the state variables when the control is approximated by piecewise constant functions. Moreover, we derive L^∞-error estimates for both the control variable and the state variables when the control is discretized by piecewise linear functions. Finally, some numerical examples are given to demonstrate the theoretical results.

AMS subject classifications: 49J20, 65N30

Key words: Semilinear elliptic equations, optimal control problems, superconvergence, L^∞-error estimates, mixed finite element methods, postprocessing.

1. Introduction

The finite element approximation plays an important role in the numerical treatment of optimal control problems. There have been extensive studies in convergence and superconvergence of finite element approximations for optimal control problems, (see, e.g., [1, 6, 11–15, 20–24]). A systematic introduction of finite element methods for PDEs and optimal control problems can be found in, (e.g., [8, 17]). Note that all the above papers aim at the standard finite element methods for optimal controls.

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Compared with standard finite element methods, the mixed finite methods have many advantages. When the objective functional contains gradient of the state variable, we will firstly choose the mixed finite element methods. We have done some works on priori error estimates and superconvergence properties of mixed finite elements for optimal control problems [3–5]. In [4], we used the postprocessing projection operator, which was defined by Meyer and Rösch (see [20]) to prove a quadratic superconvergence of the control by mixed finite element methods. Recently, we derived error estimates and superconvergence of mixed methods for convex optimal control problems in [5]. But in that paper, the convergence order is $h^{\frac{3}{2}}$ since the analysis was restricted by the low regularity of the control.

The goal of this paper is to derive the superconvergence property and the L^∞-error estimates of mixed finite element approximation for a semilinear elliptic control problem with an integral constraint. Firstly, when the control is approximated by piecewise constant functions, we derive the superconvergence property between average L^2 projection and the approximation of the control variable, the convergence order is h^2 instead of $h^{\frac{3}{2}}$ in [5], which is caused by the different admissible set. Then, after solving a fully discretized optimal control problem, a control $\hat{u}$ is calculated by the projection of the adjoint state z_h in a postprocessing step. Although the approximation of the discretized solution is only of order h, we will show that this postprocessing step improves the convergence order to h^2. We also derive the L^∞-error estimates for both the control variable and the state variables when the control variable is discretized by piecewise linear functions. Finally, we present two numerical experiments to demonstrate the practical side of the theoretical results about superconvergence and L^∞-error estimates.

We consider the following semilinear optimal control problems for the state variables $\boldsymbol{p}$, y, and the control u with an integral constraint:

$$\min_{u\in U_{ad}}\left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\right\} \tag{1.1}$$

subject to the state equation

$$-\mathrm{div}(A(x)\mathbf{grad}y)+\phi(y)=u, \quad x\in\Omega, \tag{1.2}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{p}+\phi(y)=u, \quad \boldsymbol{p}=-A(x)\mathbf{grad}y, \quad x\in\Omega, \tag{1.3}$$

and the boundary condition

$$y=0, \quad x\in\partial\Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^2$. U_{ad} denotes the admissible set of the control variable, defined by

$$U_{ad}=\left\{u\in L^\infty(\Omega):\ \int_\Omega u\,dx\geq 0\right\}. \tag{1.5}$$