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A NEW ENERGY-CONSERVED S-FDTD SCHEME FOR MAXWELL'S EQUATIONS IN METAMATERIALS

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Abstract. In this paper, we develop a new energy-conserved S-FDTD scheme for the Maxwell's equations in metamaterials. We first derive out the new property of energy conservation of the governing equations in metamaterials, and then propose the energy-conserved S-FDTD scheme for solving the problems based on the staggered grids. We prove that the proposed scheme is energy-conserved in the discrete form and unconditionally stable. Based on the energy method, we further prove that the scheme for the Maxwell's equations in metamaterials is first order in time and second order in space. Numerical experiments are carried out to confirm the energy conservation and the convergence rates of the scheme. Moreover, numerical examples are also taken to show the propagation features of electromagnetic waves in the DNG metamaterials.

Key words. Maxwell's equations, metamaterials, energy-conserved, splitting, FDTD, convergence

1. Introduction

Metamaterials are defined as artificial engineered materials exhibiting unique or unusual properties that cannot be found in natural materials at the frequencies of interest. Metamaterials have been used in many applications, that allow going beyond some limitations encountered when using natural materials, such as microwave and optical components, interconnects for wireless telecommunications, radar and defense, nanolithography and medical imaging at sub-wavelength resolution, construction of perfect lens and so on (see, [1, 27, 25, 26, 5, 9]).

Many kinds of metamaterials, such as double-negative (DNG) materials, negative index materials (NIM), left-handed materials (LHM) and back-ward (BW) media, are constructed, developed and studied ([4, 6, 21, 31]). For examples, DNG materials mean that the permittivity and permeability of the materials are both negative at the frequencies of interest; the NIM materials refer to the fact that the materials with simultaneously negative real parts of permittivity and permeability exhibit a negative real part of the refraction index, leading to anomalous refraction properties. In these kinds of metamaterials, the periodicity is much smaller than the wavelength of the impinging electromagnetic wave. Hence they are useful and customary continuous materials.

Some study of numerical simulations for Maxwell's equations in metamaterials have carried out (see, for example, [28]). Such simulations are exclusively based on the finite-difference time-domain (FDTD) methods. Due to the constraint (i.e. the CFL stability condition) of the FDTD methods, it leads to impractical computational costs and memory requirement in broadband applications in high dimensional and large domains. Therefore, there is an urgent call for developing more efficient and reliable numerical methods for metamaterial simulations. To overcome the CFL restriction of the FDTD schemes for the standard Maxwell's equations, some

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ADI-FDTD ([22, 29, 8, 12]), S-FDTD ([7]) and EC-S-FDTD ([2]) schemes were developed for the standard Maxwell's equations. But, there are few results of the ADI-FDTD, S-FDTD schemes for solving the electromagnetic problems in metamaterials. On the theoretical analysis aspect, for problems of metamatrials, some work of finite element methods (FEM) and the discontinuous Galerkin methods (DG) were analayzed ([10, 11, 13, 14, 15, 16, 17, 18, 19, 20]). However, there is no theoretical analysis work of the FDTD, ADI-FDTD, S-FDTD schemes for the Maxwell's equations that metamaterials are involved.

In this paper, we develop the energy-conserved S-FDTD scheme for the Maxwell's equations in metamaterials by focussing on preserving physical property of energy conservation. We first derive out the new property of energy conservation of the Maxwell's equations in metamaterials. We then propose an energy-conserved splitting FDTD scheme (EC-S-FDTD) for solving the problems that metamaterials are involved. We prove that the proposed EC-S-FDTD scheme satisfies the energy-conserved identity in the discrete form and the scheme is unconditionally stable. We further analyze the error estimates of the scheme and prove strictly that the scheme is of first-order convergence in time step and second-order convergence in spatial step. In numerical experiments, we show numerically the energy conservation property and the convergence rates of the EC-S-FDTD scheme and also simulate numerically the physical phenomena of electromagnetic wave propagation in the NDG metamaterials ([24, 30, 23]). Numerical results confirm our theoretical results.

The paper is organized as follows. Section 2 introduces the Maxwell's equations in metamaterials and derives the new energy conservation identity of electromagnetic fields in metamaterials. A new energy-conserved S-FDTD scheme is proposed in Section 3. We prove the discrete energy conservation, the unconditional stability of the proposed scheme in Section 4 and analyze the convergence of the scheme in Section 5. Numerical experiments are presented in Section 6. Finally, conclusions are addressed in Section 7.

2. Maxwell's equations in metamaterials

We consider the Maxwell's equations in the DNG metamaterials with the lossy Drude model. The general Maxwell's equations are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \tag{2}$$

where $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{H}(\mathbf{x},t)$ are the electric and magnetic fields, $\mathbf{D}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ are the corresponding electric and magnetic flux densities. $\mathbf{D}$ and $\mathbf{B}$ are related to $\mathbf{E}$ and $\mathbf{H}$ through the constitutive relations:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \equiv \epsilon \mathbf{E}, \qquad \mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} \equiv \mu \mathbf{H}, \tag{3}$$

where ϵ_0 is the vacuum permittivity, μ_0 is the vacuum permeability, and $\mathbf{P}$ and $\mathbf{M}$ are the induced electric and magnetic polarizations, respectively. We use the lossy Drude polarization and magnetization models [3, 32] to describe the DNG metamaterials. In the frequency domain, the permittivity and permeability are described as:

$$\epsilon(\omega) = \epsilon_0\big(1 - \frac{\omega_{pe}^2}{\omega(\omega + i\Gamma_e)}\big), \quad \mu(\omega) = \mu_0\big(1 - \frac{\omega_{pm}^2}{\omega(\omega + i\Gamma_m)}\big), \tag{4}$$