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Spectral Method for Nonlinear Stochastic Partial Differential Equations of Elliptic Type

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Abstract. This paper is concerned with the numerical approximations of semi-linear stochastic partial differential equations of elliptic type in multi-dimensions. Convergence analysis and error estimates are presented for the numerical solutions based on the spectral method. Numerical results demonstrate the good performance of the spectral method.

AMS subject classifications: 65N35, 65N15, 65C30, 60H15

Key words: Stochastic differential equation, spectral method, error estimates, white noise.

1. Introduction

Many natural phenomena and engineering applications are described by stochastic partial differential equations (SPDEs). The study of numerical methods for approximating stochastic partial differential equations has been an active research area. Some fluid flow and other engineering related SPDEs were studied using polynomial chaos expansions in [12,19,20]. In [3,4,6,11,15], traditional finite element methods are applied to SPDEs with random coefficients. Numerical methods for SPDEs with white noise forcing terms have also been developed, analyzed, and tested by numerous authors [2,7–10,13,14,16,17].

The main purpose of this paper is to study the numerical approximations by a spectral method for nonlinear stochastic differential equations of elliptic type driven by an additive white noise:

$$\Delta u(x) - f(u(x)) = g(x) + \dot{W}(x), \quad \text{for } x \in D, \tag{1.1}$$

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with boundary condition $u(x) = 0$ for $x \in \partial D$. Here D is a bounded open set of $\mathbb{R}^d$, $g \in L^2(D)$, f is a continuous function satisfying certain regularity conditions given in Section 2, and

$$\dot{W}(x) = \frac{dW}{dx}(x)$$

is a white noise. Buckdahn and Pardoux have proved the existence and uniqueness of the weak solution for (1.1) in [5]. Besides, this solution is almost surely continuous on $\overline{D}$. In [9] Gyöngy and Martínez considered the finite difference approximation for (1.1). They converted (1.1) into an integral equation using the Green's function and obtained the convergence rates for the approximate solution under certain regularity assumptions on f.

In this paper, the white noise processes are approximated by piecewise constant random processes (as in [2] and [7]). First we introduce a rectangular partition of $D = [0,1]^d$. For each direction of $x = (x^1, x^2, \cdots, x^d) \in D$, there exists a partition $0 = x_1^j < x_2^j < \cdots < x_{N+1}^j = 1$ with $x_l^j = (l-1)h$, $\forall l \in \{1, \cdots, N+1\}$,$\forall j \in \{1, \cdots, d\}$, where $h = 1/N$. Then, $D = [0,1]^d$ is divided into disjointed cells

$$D_i = \left\{ x \in D \mid x_l^j \le x^j < x_{l+1}^j,\ \forall\, j \in \{1, \cdots, d\},\ \forall\, l \in \{1, \cdots, N\} \right\}, \quad i = 1, \cdots, N^d,$$

and $|D_i| = 1/N^d$. A reasonable approximation to $\frac{dW}{dx}(x)$ is

$$\frac{d\widehat{W}}{dx}(x) = N^d \sum_{i=1}^{N^d} \eta_i \sqrt{|D_i|}\chi_i(x), \tag{1.2}$$

where

$$\sqrt{|D_i|}\eta_i = \int_{D_i} dW(x), \quad \text{for } i = 1, \cdots, N,$$

i.e., $\eta_i \in N(0,1)$ are independent identically distributed random variables, and

$$\chi_i(x) = \begin{cases} 1, & \text{if } x \in D_i, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\widehat{u}(x)$ be the approximation of $u(x)$ given by

$$\Delta \widehat{u}(x) - f(\widehat{u}(x)) = g(x) + \dot{\widehat{W}}(x), \quad \text{for } x \in D, \tag{1.3}$$

with boundary condition $\widehat{u} = 0$ for $x \in \partial D$.

The key to the error analysis of finite element and spectral methods is the regularity of the solution of the underlying SPDE. Unfortunately, As shown in [2] and other literatures, the required regularity conditions for the standard error estimates of the finite element method are not satisfied for the problem (1.1). With discretized process, we will show that the solution $\widehat{u}$ of the corresponding SPDE (1.3) has a certain regularity, which allows to obtain an error estimate.