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ON CONVERGENCE OF THE STREAMLINE DIFFUSION AND DISCONTINUOUS GALERKIN METHODS FOR THE MULTI-DIMENSIONAL FERMI PENCIL BEAM EQUATION

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Abstract. We derive error estimates in the L_2 norms, for the streamline diffusion (SD) and discontinuous Galerkin (DG) finite element methods for steady state, energy dependent, Fermi equation in three space dimensions. These estimates yield optimal convergence rates due to the maximal available regularity of the exact solution. Here our focus is on theoretical aspects of the h and hp approximations in both SD and DG settings.

Key words. Fermi equation, particle beam, streamline diffusion, discontinuous Galerkin, stability, convergence

1. Introduction

We study approximate solutions for the three-dimensional Fermi equation using streamline diffusion (SD) and discontinuous Galerkin (DG) finite element methods. We prove stability estimates and derive optimal convergence rates for the current function. This work extends the results in [2]-[3] to the multidimensional case, and includes the *hp* approach. The physical problem has diverse applications in, e.g. astrophysics, material science, electron microscopy, radiation therapy, etc. We shall consider a pencil beam of particles normally incident on a slab of finite thickness, entering the slab at a single point, e.g. $(0, 0, 0)$, in the direction of positive x-axis.

Fermi equation is a convection-diffusion equation, obtained as an asymptotic limit of the Fokker-Planck equation as the *transport cross-section* (σ_{tr}) gets smaller, see [7]. The equation is *degenerate* in both convection and diffusion in the sense that drift and diffusion are taking place in, physically, different domains, and the problem is *convection dominated*. Further, the associated boundary conditions are in the form of product of δ functions, which are not suitable for L_2-estimates. Therefore, we consider model problems with data smoother than Dirac δ-function.

Fermi equation has closed form solutions for σ_{tr} being a constant or a function of only x. In the present setting the direction of penetration of the beam, x, may also be interpreted as the direction of a *hypothetic* time variable.

The SD-method is obtained modifying the weak form by adding a multiple of the "drift-terms" in the equation to the test function. This yields artificial diffusion added only in the streamlines direction (motivating for the name: *the streamline diffusion method*) which improves stability in the characteristic direction so that internal layers are not smeared out while the added diffusion removes oscillations near boundary layers. The oscillations merge from the lack of stability of standard Galerkin for convection dominated problems, see, e.g. [14]. While SD may have discontinuities in x-direction only, the DG method allows jump discontinuities across interelement boundaries in order to count for the local effects. We study both h and hp versions of SD and DG methods. A semi-streamline diffusion for Fermi

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equation has been implemented in [3]. The hp version is considered in a general setting for a Vlasov-Poisson-Fokker-Planck system in [5].

An outline of this paper is as follows: In Section 2, we introduce the model problem. Section 3 is devoted to the stability estimates and convergence analysis for the h and hp streamline diffusion approximations of the Fermi equation. Section 4 is the discontinuous Galerkin counterpart of Section 3, counting for local properties.

2. Model Problem

We consider a model problem for three dimensional Fermi equation on a bounded polygonal domains $\Omega_{\mathbf{x}} \subset \mathbb{R}^3$, $\mathbf{x} = (x, y, z) =: (x, x_\perp)$, with velocities $v \in \Omega_v \subset \mathbb{R}^2$:

$$(2.1)\quad \begin{cases} \frac{\partial f}{\partial x} + v \cdot \nabla_\perp f = \frac{\sigma_{tr}}{2}(\Delta_v f), & \text{in } (0, L] \times \Omega =: Q_L, \\ f(0, x_\perp, v) = f_0(x_\perp, v), & \text{in } \Omega = \Omega_{x_\perp} \times \Omega_v, \\ f(x, x_\perp, v) = 0, & \text{in } (0, L] \times ([\Gamma_v^- \times \Omega_v] \cup [\Omega_{x_\perp} \times \partial\Omega_v]), \end{cases}$$

where $f_0 \in L_2(\Omega)$, and for each $v \in \Omega_v$, the outflow boundary is given by

$$(2.2)\quad \Gamma_v^- = \{x_\perp \in \partial\Omega_{x_\perp} : \mathbf{n}(x_\perp).v < 0\}.$$

Here $\Omega_\perp = \{(y, z)\}$, $\mathbf{n}(x_\perp)$ is the outward unit normal to $\partial\Omega_{x_\perp}$ at the point $x_\perp = (y, z) \in \partial\Omega_{x_\perp}$, $v = (v_1, v_2)$, $\nabla_\perp = (\frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ and $\sigma_{tr} = \sigma_{tr}(x, y, z)$.

2.1. Notations and preliminaries.

Let $T_h^{x_\perp} = \{\tau_{x_\perp}\}$ and $T_h^v = \{\tau_v\}$ be finite element subdivisions of $\Omega_{x_\perp}$ and Ω_v, into the elements $\tau^{x_\perp}$ and τ^v, respectively. Thus, $T_h = T_h^{x_\perp} \times T_h^v$ will be a subdivision of $\Omega = \Omega_{x_\perp} \times \Omega_v$ with elements $\{\tau_{x_\perp} \times \tau_v\} = \{\tau\}$. Consider a partition $\mathcal{T}_h :\ 0 = x_0 < x_1 < \ldots < x_M = L$ of the interval $I = (0, L]$ into subintervals $I_m = (x_{m-1}, x_m],\ m = 1, ..., M$, and let $\mathcal{C}_h$ be the corresponding subdivision of $Q_L := (0, L] \times \Omega$ into elements $K = I_m \times \tau$ with the mesh size $h_K = \text{diam } K$. We assume that each $K \in \mathcal{C}_h$ is the image under a family of bijective affine maps $\{F_K\}$ of a fixed standard element $\hat{K}$ into K, where $\hat{K}$ is either the open unit simplex or the open unit hypercube in $\mathbb{R}^5$ (in the hp-analysis, $\hat{K}$ is the open unit hypercube in $\mathbb{R}^5$). Let $P_p(K)$ be the set of all polynomials of degree $\leq p$ on K; in $x, x_\perp$ and v, and define the finite element space

$$(2.3)\quad V_h = \{g \in \widetilde{\mathcal{H}}_0 : g \circ F_K \in P_p(\hat{K});\ \forall K \in \mathcal{C}_h\}, \qquad \text{where}$$

$$(2.4)\quad \widetilde{\mathcal{H}}_0 = \prod_{m=1}^{M} H_0^1(S_m), \qquad S_k = I_k \times \Omega, \qquad k = 1, \cdots, M, \qquad \text{with}$$

$$(2.5)\quad H_0^1(S_m) = \{g \in H^1(S_m) : g \equiv 0 \quad \text{on } \partial\Omega_v\}.$$

For piecewise polynomials w_i defined on the triangulation $\mathcal{C}'_h = \{K\}$ with $\mathcal{C}'_h \subset \mathcal{C}_h$ and for D_i being some differential operators, we use the notation,

$$(2.6)\quad (D_1 w_1, D_2 w_2)_{Q'} = \sum_{K \in \mathcal{C}'_h} (D_1 w_1, D_2 w_2)_K, \qquad Q' = \bigcup_{K \in \mathcal{C}'_h} K,$$

where $(.,.)_Q$ is the $L_2(Q)$ scalar product and $\|.\|_Q$ is the corresponding $L_2(Q)$-norm. Further, for $m = 1, 2, \ldots, M$, $\beta = (v, \mathbf{0})$, $\mathbf{n} = (\mathbf{n}_{x_\perp}, \mathbf{n}_v)$ and with $\Gamma = \partial(\Omega_{x_\perp} \times \Omega_v)$,

$$(2.7)\quad \begin{array}{ll} (f, g)_m = (f, g)_{S_m}, & \|g\|_m^2 = (g, g)_m, \\ \langle f, g\rangle_m = (f(x_m, ., .), g(x_m, ., .))_\Omega, & |g|_m^2 = \langle g, g\rangle_m, \\ \langle f, g\rangle_{\Gamma^-} = \int_{\Gamma^-} fg(\beta \cdot \mathbf{n}) ds, & \langle f, g\rangle_{\Gamma_m^-} = \int_{I_m} \langle f, g\rangle_{\Gamma^-} ds, \\ \langle f, g\rangle_{\Gamma_I^-} = \int_I \langle f, g\rangle_{\Gamma^-} ds, & \Gamma^- = \{(x_\perp, v) \in \Gamma : \beta \cdot \mathbf{n} < 0\}, \end{array}$$