INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 8, Number 1, Pages 28–47

THE MULTISCALE DISCONTINUOUS GALERKIN METHOD FOR SOLVING A CLASS OF SECOND ORDER ELLIPTIC PROBLEMS WITH ROUGH COEFFICIENTS

WEI WANG, JOHNNY GUZMÁN, AND CHI-WANG SHU

Abstract. We develop a multiscale discontinuous Galerkin (DG) method for solving a class of second order elliptic problems with rough coefficients. The main ingredient of this method is to use a non-polynomial multiscale approximation space in the DG method to capture the multiscale solutions using coarse meshes without resolving the fine scale structure of the solution. Theoretical proofs and numerical examples are presented in both one and two dimensions. For one-dimensional problems, optimal error estimates and numerical examples are shown for arbitrary order approximations. For two-dimensional problems, numerical results are presented by the high order multiscale DG method, but the error estimate is proven only for the second order method.

Key words. multiscale discontinuous Galerkin method, rough coefficients

1. Introduction

In this paper, we consider solving a class of second order elliptic boundary value problems with highly oscillatory coefficients. Such equations arise in, e.g. composite materials and porous media. The solution oscillates rapidly and requires a very refined mesh to resolve. It is numerically difficult for traditional numerical methods to solve such problems due to the tremendous amount of computer memory and CPU time. Recently developed multiscale finite element methods [3, 13, 2, 15, 16, 11, 6, 23] provide an idea of constructing multiscale bases which are adapted to the local properties of the differential operators, allowing adequate resolution on a coarser mesh.

In particular, we are interested in the second order elliptic boundary value problems

$$-\nabla \cdot (A(\mathbf{x})\nabla u) = f(\mathbf{x}) \quad \text{in } \Omega \tag{1}$$

with the boundary condition

$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a rectangular domain, f is a function in $L^2(\Omega)$ and $A(\mathbf{x})$ is the coefficient matrix containing small scales.

In applications, Eq. (1) is the pressure equation in modeling two phase flow in porous media (see [17, 15, 6]), with u and $A(\mathbf{x})$ interpreted as the pressure and the relative permeability tensor. Especially when the stochastic permeabilities are upscaled, $A(\mathbf{x})$ is a diagonal tensor. Eq. (1) is also the equation of steady state heat (electrical) conduction through a composite material, with $A(\mathbf{x})$ and u interpreted as the thermal (electric) conductivity and temperature (electric potential) (see [15]).

In the one-dimensional case, $A(\mathbf{x}) = a(x)$ and the equation becomes

$$-(a(x)u_x)_x = f(x). \tag{2}$$

Received by the editors April 21, 2010.
2000 *Mathematics Subject Classification.* 65N30.
Research partially supported by NSF grant DMS-0914596, DOE grant DE-FG02-08ER25863 and NSF grant DMS-0809086.

In the two-dimensional case, we consider A with the following special form

$$A(\mathbf{x}) = \begin{pmatrix} a(x) & 0 \\ 0 & b(y) \end{pmatrix},$$

and the two-dimensional equation is

$$-(a(x)u_x)_x - (b(y)u_y)_y = f(x,y). \tag{3}$$

The typical situation in multiscale modeling, has $a(x) = a^\varepsilon(x,\varepsilon)$ and $b(y) = b^\varepsilon(y,\varepsilon)$ being oscillatory functions involving a small scale ε. We do not need the assumption of any periodicity of a^ε and b^ε and we do not assume any scale separation. The coefficients as well as the solution u can then have a continuum scale spectrum from $O(\varepsilon)$ to $O(1)$. We only assume $a(x)$ and $b(y)$ belong to $L^\infty(\Omega)$ and satisfy

$$0 < \alpha \le a(x), b(y) \le \beta < \infty \tag{4}$$

for any $(x,y) \in \Omega$, where α and β are constants independent of ε. If the coefficient $a(x)$ is rough, then the solution u to (2) will also be rough; to be more specific, we will in general have

$$||a||_{H^1(\Omega)} \to \infty, \qquad ||u||_{H^2(\Omega)} \to \infty, \qquad \text{as } \varepsilon \to 0.$$

Thus, u is not uniformly bounded with respect to ε in $H^2(\Omega)$ or in $H^{1+\delta}(\Omega)$ for any $\delta > 0$.

Notice that we are considering the special class (3) of two-dimensional problems for the convenience of explicitly constructed multiscale bases, thereby making the multiscale algorithm efficient. This special class of multiscale problems does have important applications in, e.g. two-dimensional semi-conductor quantum devices (for application of such device models in one-dimension, see [20]), in which there is a specific direction of oscillation in the coefficients at each location in space and time. We remark that the proposed numerical method can also be applied to cases with more general two-dimensional coefficients, at the price of having to numerically constructing the multiscale bases.

As early as in the 60s, Tikhonov and Samarskii [18] (see also [14]) already designed a simple 3-point finite difference scheme utilizing harmonic averages and the special solution structure of (2). In particular, the scheme in [14] can give exact solutions to the one-dimensional problem (2) at the grid points. In [3, 2], Babuška et al. proposed an approach to this kind of problems based on continuous (or nonconforming) finite element methods. In [3] theoretical proofs were provided for the one-dimensional case and arbitrary order approximation. The two-dimensional case was considered in [2], where only second-order accurate elements were considered (piecewise linear elements if A is constant). One of the difficulties in using higher order elements in multi-dimensions for the continuous Galerkin method is to make the multi-scale spaces conforming. Compared to continuous finite element methods, discontinuous Galerkin (DG) methods do not enforce continuity at the element interfaces, thus providing an easy way to construct multiscale basis in higher dimensions with high-order elements. Of course, there is a price to pay for the DG multiscale method for this flexibility: we must carefully analyze the errors associated with these discontinuities across element interfaces, to obtain high order error estimate. This is done for the arbitrary high order scheme in one dimension and for the second order scheme in two dimension in this paper. Numerical evidence indicates that our multiscale DG scheme can achieve higher than second order accuracy in two dimensions, as shown in this paper, although a proof is not available at this time.