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ADI FINITE DIFFERENCE SCHEMES FOR OPTION PRICING IN THE HESTON MODEL WITH CORRELATION

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Abstract. This paper deals with the numerical solution of the Heston partial differential equation (PDE) that plays an important role in financial option pricing theory, Heston (1993). A feature of this time-dependent, two-dimensional convection-diffusion-reaction equation is the presence of a mixed spatial-derivative term, which stems from the correlation between the two underlying stochastic processes for the asset price and its variance.

Semi-discretization of the Heston PDE, using finite difference schemes on non-uniform grids, gives rise to large systems of stiff ordinary differential equations. For the effective numerical solution of these systems, standard implicit time-stepping methods are often not suitable anymore, and tailored time-discretization methods are required. In the present paper, we investigate four splitting schemes of the Alternating Direction Implicit (ADI) type: the Douglas scheme, the Craig–Sneyd scheme, the Modified Craig–Sneyd scheme, and the Hundsdorfer–Verwer scheme, each of which contains a free parameter.

ADI schemes were not originally developed to deal with mixed spatial-derivative terms. Accordingly, we first discuss the adaptation of the above four ADI schemes to the Heston PDE. Subsequently, we present various numerical examples with realistic data sets from the literature, where we consider European call options as well as down-and-out barrier options. Combined with ample theoretical stability results for ADI schemes that have recently been obtained in In 't Hout & Welfert (2007, 2009) we arrive at three ADI schemes that all prove to be very effective in the numerical solution of the Heston PDE with a mixed derivative term. It is expected that these schemes will be useful also for general two-dimensional convection-diffusion-reaction equations with mixed derivative terms.

Key Words. Initial-boundary value problems, convection-diffusion equations, mixed derivatives, Heston model, option pricing, method-of-lines, finite difference methods, ADI splitting schemes.

1. Introduction

In the Heston model, values of options are given by a time-dependent partial differential equation (PDE) that is supplemented with initial and boundary conditions [7, 14, 22, 24]. The Heston PDE constitutes an important two-dimensional extension to the celebrated, one-dimensional, Black–Scholes PDE. Contrary to the

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Black–Scholes model, however, to date in the Heston model no closed-form analytical formulas have been found for any but the simplest options, and therefore numerical techniques are applied.

A well-known and versatile strategy for the numerical solution of initial-boundary value problems for multi-dimensional PDEs is the method-of-lines approach. In this approach, the PDE is first discretized in the spatial variables, yielding large systems of stiff ordinary differential equations. These, so-called, semi-discrete systems are subsequently solved by applying a suitable numerical time-stepping method. Due to the large size and the multi-dimensional structure of the obtained semi-discrete systems, standard time-stepping methods, such as the popular Crank–Nicolson scheme (trapezoidal rule), are often not effective anymore, and tailored time discretization methods are required.

For the numerical solution of the semi-discrete Heston PDE we shall study in this paper splitting schemes of the Alternating Direction Implicit (ADI) type. In the past decades, ADI schemes have been successful already in many application areas. A main and distinctive feature of the Heston PDE, however, is the presence of a mixed spatial-derivative term, stemming from the correlation between the two underlying stochastic processes for the asset price and its variance. It is well-known that ADI schemes were not originally developed to deal with such terms. In the present paper, we will investigate the adaptation of several important ADI schemes to the numerical solution of the Heston PDE with arbitrary correlation factor $\rho \in [-1, 1]$. As test cases we will consider European call options and down-and-out barrier options. Through various numerical examples with realistic data sets from the literature, combined with ample theoretical stability results that have recently been obtained, we arrive at three ADI schemes that all prove to be very effective in the numerical solution of the Heston PDE with a mixed derivative term. It is expected that these schemes will be useful also for general two-dimensional convection-diffusion-reaction equations with mixed derivative terms.

An outline of our paper is as follows.

Section 2 discusses the Heston PDE and its numerical discretization. In Section 2.1 we formulate the Heston PDE together with initial and boundary conditions for European call options. In Section 2.2 we describe a finite difference discretization of the Heston PDE. A non-uniform spatial grid is used to capture the important region around the strike. In Section 2.3 we formulate the ADI type schemes under consideration in this paper for the semi-discrete Heston PDE with a mixed derivative term: the Douglas scheme, the Craig–Sneyd scheme, the Modified Craig–Sneyd scheme, and the Hundsdorfer–Verwer scheme. Each of these contains a free parameter θ. We discuss the different origins of the four schemes and review theoretical stability results that were recently obtained in [9, 10] concerning their application to multi-dimensional convection-diffusion equations with mixed derivative terms.

Section 3 contains extensive numerical experiments. In Section 3.1 we study the accuracy of our finite difference discretization in various examples of parameter sets for the Heston model obtained from the literature. Here the availability of Heston's analytical pricing formula for European call options makes an actual computation of the global spatial errors possible. In Section 3.2 we perform numerical experiments with all the ADI schemes above, where we analyze the behavior of the global temporal errors for each example introduced in Section 3.1. As an alternative method, we also consider a Runge–Kutta–Chebyshev scheme. In Section 3.3 we discuss numerical experiments for down-and-out call options. For these exotic