Commun. Comput. Phys.
doi: 10.4208/cicp.231111.090312a

Vol. **13**, No. 2, pp. 559-582
February 2013

Adaptive *hp*-Finite Element Computations for Time-Harmonic Maxwell's Equations

Xue Jiang[1], Linbo Zhang[1] and Weiying Zheng[1,*]

[1] *LSEC, Institute of Computational Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China.*

Received 23 November 2011; Accepted (in revised version) 9 March 2012

Available online 3 August 2012

Abstract. In this paper, hp-adaptive finite element methods are studied for time-harmonic Maxwell's equations. We propose the parallel hp-adaptive algorithms on conforming unstructured tetrahedral meshes based on residual-based a posteriori error estimates. Extensive numerical experiments are reported to investigate the efficiency of the hp-adaptive methods for point singularities, edge singularities, and an engineering benchmark problem of Maxwell's equations. The hp-adaptive methods show much better performance than the h-adaptive method.

AMS subject classifications: 65M30, 78A45

Key words: hp-adaptive finite element method, Maxwell's equations, eddy current problem, a posteriori error estimate.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded polygonal domain with Lipschitz continuous boundary $\Gamma = \partial\Omega$. Given a current density $\boldsymbol{f}$ with $\operatorname{div} \boldsymbol{f} = 0$, we seek a solution $\boldsymbol{u}$ which satisfies the following time-harmonic Maxwell's equations (cf. e.g. [33, Chapter 1])

$$\mathbf{curl}(\mu_r^{-1}\mathbf{curl}\,\boldsymbol{u}) - \kappa^2\alpha\boldsymbol{u} = \boldsymbol{f}, \quad \text{in } \Omega, \tag{1.1a}$$

$$\mu_r^{-1}\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = 0, \qquad \text{on } \Gamma, \tag{1.1b}$$

where $\mu_r \geq 1$ is the relative magnetic permeability, $\kappa > 0$ is the constant wave number, and α is the complex relative dielectric coefficient. Usually $\boldsymbol{u}$ stands for the electric field or the magnetic vector potential. In this paper, we are interested in two kinds of applications of

*Corresponding author. *Email addresses:* `jxue@lsec.cc.ac.cn` (X. Jiang), `zlb@lsec.cc.ac.cn` (L. Zhang), `zwy@lsec.cc.ac.cn` (W. Zheng)

(1.1). For $\alpha = \varepsilon_r - \mathbf{i}\sigma(\omega\varepsilon_0)^{-1}$, (1.1) describes the electromagnetic field at moderate or high frequencies. For $\alpha = -\mathbf{i}\sigma(\omega\varepsilon_0)^{-1}$, (1.1) is the eddy current model which approximates the Maxwell equations at very low frequency [3]. Here $\varepsilon_r \geq 1$ is the physical relative dielectric coefficient, $\varepsilon_0 > 0$ is the dielectric coefficient in the empty space, $\sigma \geq 0$ is the electric conductivity, and $\omega > 0$ is the constant angular frequency. For the eddy current model, we assume that Γ is the truncated boundary of $\mathbb{R}^3$ and Ω is *simply connected*. The homogeneity of the boundary condition in (1.1b) is not essential for our analysis but simplifies our proofs.

In this paper, we study the hp-adaptive finite element method for (1.1) on tetrahedral meshes. Since the pioneering work of Babuška and Rheinboldt [8], the self-adaptive finite element method based on a posteriori error estimates has been studied over thirty years. It has become one of the most popular methods in the numerical solution of partial differential equations and in scientific and engineering computing. As for the h-adaptive method which reduces the error by local mesh refinements, great successes have been achieved in the study of a posteriori error analysis (cf. e.g. [11, 16, 40]), mesh refinement algorithms (cf. e.g. [4, 27, 41]), and convergence and optimal complexity (cf. e.g. [22, 26, 34]). Recently, Schöberl and coauthors studied h-type a posteriori error estimates for Maxwell's equations. In [14], Braess and Schöberl proposed the equilibrated residual error estimator for edge elements with unit reliability constant. In [39], Schöberl proved the reliability of the residual-based a posteriori error estimate for Maxwell's equations via commuting quasi-interpolation operators. Using the idea of error equidistribution, the h-adaptive method based on a posteriori error estimates could yield a quasi-optimal approximation with algebraic convergence rate

$$\eta_h \approx C N_h^{-p/d}, \tag{1.2}$$

where η_h is the a posteriori error estimate, d is the spatial dimension, p is the order of the finite element method, and N_h is the number of degrees of freedom. However, due to the singularity of the solution, the quasi-optimality (1.2) will degenerate for higher-order finite elements since the constant C may blow up with increasing p.

The hp-adaptive finite element method reduces the error by both local mesh refinements and local increase of polynomial degrees. It is more efficient than the pure h-adaptive and p-adaptive methods and could reduce the error exponentially. For example, the optimal convergence rate of the hp-adaptive method is

$$\eta_{hp} \approx C e^{-\delta N_{hp}^{1/3}} \tag{1.3}$$

for two dimensional elliptic problems [23], and is also conjectured to be

$$\eta_{hp} \approx C e^{-\delta N_{hp}^{1/5}} \tag{1.4}$$

for three dimensional elliptic problems [6], where C,δ are positive constants independent of h and p, η_{hp} is the a posteriori error estimate for the hp-adaptive method, and N_{hp} is the