Commun. Comput. Phys.
doi: 10.4208/cicp.020611.090312a

Vol. **13**, No. 4, pp. 1013-1044
April 2013

Differential Formulation of Discontinuous Galerkin and Related Methods for the Navier-Stokes Equations

Haiyang Gao[1,*], Z. J. Wang[1] and H. T. Huynh[2]

[1] *Department of Aerospace Engineering and CFD Center, Iowa State University, 2271 Howe Hall Ames, IA 50011, USA.*
[2] *NASA Glenn Research Center, MS 5-11, Cleveland, OH 44135, USA.*

Received 2 June 2011; Accepted (in revised version) 9 March 2012

Available online 21 September 2012

Abstract. A new approach to high-order accuracy for the numerical solution of conservation laws introduced by Huynh and extended to simplexes by Wang and Gao is renamed CPR (correction procedure or collocation penalty via reconstruction). The CPR approach employs the differential form of the equation and accounts for the jumps in flux values at the cell boundaries by a correction procedure. In addition to being simple and economical, it unifies several existing methods including discontinuous Galerkin, staggered grid, spectral volume, and spectral difference. To discretize the diffusion terms, we use the BR2 (Bassi and Rebay), interior penalty, compact DG (CDG), and I-continuous approaches. The first three of these approaches, originally derived using the integral formulation, were recast here in the CPR framework, whereas the I-continuous scheme, originally derived for a quadrilateral mesh, was extended to a triangular mesh. Fourier stability and accuracy analyses for these schemes on quadrilateral and triangular meshes are carried out. Finally, results for the Navier-Stokes equations are shown to compare the various schemes as well as to demonstrate the capability of the CPR approach.

AMS subject classifications: 76N15

Key words: Discontinuous Galerkin, lifting collocation penalty, flux reconstruction, Navier-Stokes equations, correction procedure via reconstruction, unstructured hybrid grids.

1 Introduction

Second-order methods are currently popular in fluid flow simulations. For many important problems such as computational aeroacoustics, vortex-dominant flows, and large

*Corresponding author. *Email addresses:* `hgao@iastate.edu` (H. Gao), `zjw@iastate.edu` (Z. J. Wang), `hung.t.huynh@nasa.gov` (H. T. Huynh)

eddy and direct numerical simulation of turbulent flows, the number of grid points required by a second-order scheme is often beyond the capacity of current computers. For these problems, high-order methods hold the promise of accurate solutions with a manageable number of grid points. Numerous high-order methods have been developed in the last two decades. Here, we focus only on those that employ a polynomial to approximate the solution in each cell or element, and the polynomials collectively form a function which is discontinuous across cell boundaries. Commonly used methods of this type include discontinuous Galerkin (DG) [2, 3, 5–7], staggered-grid (SG) [15], spectral volume (SV) [24–27], and spectral difference (SD) [17–19]. Among these, DG and SV are usually formulated via the integral form of the equation, whereas SG and SD, the differential one. From an algorithm perspective, the difference among these methods lies in the definition of the degrees of freedom (DOFs), which determine the polynomial in each cell, and how these DOFs are updated.

High-order methods for conservation laws discussed above deal with the first derivative. Diffusion problems (viscous flows) involve the second derivative. There are many ways to extend a method of estimating the first derivative to the second; Arnold et al. analyzed several of them in [1]. Here, we restrict ourselves to approaches of compact stencil: the second derivative estimate in an element involves data of only that element and the immediate face neighbors. Such approaches have several advantages: the associated boundary conditions are simpler, the coding is easier, and the implicit systems are smaller. The four schemes of compact stencil employed are BR2 (Bassi and Rebay) [4], compact DG or CDG [20], interior penalty [9,12], and I-continuous (the value and derivative are continuous across the interface) [14]. The BR2 scheme, an improvement of the non-compact BR1 [2], is the first successful approach of this type for the Navier-Stokes equations. The CDG scheme is a modification of the local DG or LDG [8] to obtain compactness for an unstructured mesh. The interior penalty scheme is employed here with a penalty coefficient using correction function [14]. The I-continuous approach is highly accurate for linear problems on a quadrilateral mesh. Nicknamed ”poor man's recovery”, it can be considered as an approximation to the recovery approach of Van Leer and Nomura [23]. (The recovery approach is beyond the scope of this paper since, although it is more accurate than the schemes discussed here based on Fourier analysis [14], is more complex and costly.)

For conservation laws, Huynh (2007) [13] introduced an approach to high-order accuracy called flux reconstruction (FR). The approach solves the equations in differential form. It evaluates the first derivative of a discontinuous piecewise polynomial function by employing the straightforward derivative estimate together with a correction which accounts for the jumps at the interfaces. The FR framework unifies several existing methods: with appropriate correction terms, it recovers DG, SG, SV, SD methods. This framework was extended to diffusion problems using quadrilateral meshes in [14], where several existing schemes for diffusion were recast and analyzed. Wang and Gao (2009) [28] extended the FR idea to 2D triangular and mixed meshes with the lifting collocation penalty (LCP) formulation. The LCP method was applied to solve the Euler and