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REVIEW ARTICLE

Local Discontinuous Galerkin Methods for High-Order Time-Dependent Partial Differential Equations

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Abstract. Discontinuous Galerkin (DG) methods are a class of finite element methods using discontinuous basis functions, which are usually chosen as piecewise polynomials. Since the basis functions can be discontinuous, these methods have the flexibility which is not shared by typical finite element methods, such as the allowance of arbitrary triangulation with hanging nodes, less restriction in changing the polynomial degrees in each element independent of that in the neighbors (p adaptivity), and local data structure and the resulting high parallel efficiency. In this paper, we give a general review of the local DG (LDG) methods for solving high-order time-dependent partial differential equations (PDEs). The important ingredient of the design of LDG schemes, namely the adequate choice of numerical fluxes, is highlighted. Some of the applications of the LDG methods for high-order time-dependent PDEs are also be discussed.

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1 Overview

1.1 Brief introduction of the discontinuous Galerkin method

The discontinuous Galerkin (DG) method that we discuss in this paper is a class of finite element methods using a discontinuous piecewise polynomial space for the numerical solution and the test functions in the spatial variables, coupled with explicit or implicit nonlinearly stable high order time discretization. These methods have found their way into the main stream of computational fluid dynamics and other areas of applications.

The first DG method was introduced in 1973 by Reed and Hill [77], in the framework of neutron transport, i.e. a time independent linear hyperbolic equation. It was later developed for solving nonlinear hyperbolic conservation laws with first derivatives by Cockburn et al. in a series of papers [29,35,37,39], in which they have established a framework to easily solve nonlinear time dependent problems, such as the Euler equations in compressible gas dynamics, using explicit, nonlinearly stable high order Runge-Kutta time discretizations [85] and DG discretization in space with exact or approximate Riemann solvers as interface fluxes and total variation bounded (TVB) nonlinear limiters [82] to achieve non-oscillatory properties for strong shocks.

Since the basis functions can be discontinuous, the DG methods have certain flexibility and advantage, such as,

- It can be easily designed for any order of accuracy. In fact, the order of accuracy can be locally determined in each cell.
- It is easy to handle complicated geometry and boundary conditions. It can be used on arbitrary triangulations, even those with hanging nodes.
- It is local in data communications. The evolution of the solution in each cell needs to communicate only with its immediate neighbors, regardless of the order of accuracy. The methods have high parallel efficiency, usually more than 99% for a fixed mesh, and more than 80% for a dynamic load balancing with adaptive meshes which change often during time evolution, see, e.g. [12,78].
- There is provable cell entropy inequality and L^2 stability, for arbitrary scalar equations in any spatial dimension and any triangulation, for any order of accuracy, without limiters [60].
- It is at least $(k+\frac{1}{2})$-th order accurate, and often $(k+1)$-th order accurate in L^2 norm for smooth solutions when piecewise polynomials of degree k are used, regardless of the structure of the meshes.
- It is flexible to h-p adaptivity. A very good example to illustrate the capability of the DG method in h-p adaptivity, efficiency in parallel dynamic load balancing, and