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Explicit Symplectic Methods for the Nonlinear Schrödinger Equation

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Abstract. By performing a particular spatial discretization to the nonlinear Schrödinger equation (NLSE), we obtain a non-integrable Hamiltonian system which can be decomposed into three integrable parts (L-L-N splitting). We integrate each part by calculating its phase flow, and develop explicit symplectic integrators of different orders for the original Hamiltonian by composing the phase flows. A 2nd-order reversible constructed symplectic scheme is employed to simulate solitons motion and invariants behavior of the NLSE. The simulation results are compared with a 3rd-order non-symplectic implicit Runge-Kutta method, and the convergence of the formal energy of this symplectic integrator is also verified. The numerical results indicate that the explicit symplectic scheme obtained via L-L-N splitting is an effective numerical tool for solving the NLSE.

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Key words: Explicit symplectic method, L-L-N splitting, nonlinear Schrödinger equation.

1 Introduction

The nonlinear Schrödinger equation (NLSE) has been central to a variety of areas in mathematical physics for almost four decades. It is an equation for a complex field $W(x,t)$ of the following form along with initial condition:

$$\begin{cases} iW_t+W_{xx}+a|W|^2W=0, \\ W(x,0)=W_0(x), \end{cases} \tag{1.1}$$

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where $x \in \mathbb{R}$ and a is a constant parameter. The initial condition $W_0(x)$ conducts the motion, for instance, some $W_0(x)$'s with $W_0(\pm\infty)=0$ induce bright solitons motion; some with $|W_0(\pm\infty)| \neq 0$ lead to dark solitons motion; and periodic $W_0(x)$'s may result in periodic motion [9,22,23]. The NLSE (1.1) has an infinite number of conserved quantities [44] such as the charge, the moment, the energy, etc. We present the first six as follows:

$$F_1=\int_{-\infty}^{+\infty}|W|^2dx, \quad F_2=\int_{-\infty}^{+\infty}\Big\{W\frac{d\widetilde{W}}{dx}-\widetilde{W}\frac{dW}{dx}\Big\}dx, \tag{1.2a}$$

$$F_3=\int_{-\infty}^{+\infty}\Big\{2\left|\frac{dW}{dx}\right|^2-a|W|^4\Big\}dx, \tag{1.2b}$$

$$F_4=\int_{-\infty}^{+\infty}\Big\{2\frac{d\widetilde{W}}{dx}\frac{d^2W}{dx^2}-3a|W|^2\widetilde{W}\frac{dW}{dx}\Big\}dx, \tag{1.2c}$$

$$F_5=\int_{-\infty}^{+\infty}\Big\{2\left|\frac{d^2W}{dx^2}\right|^2-6a|W|^2\left|\frac{dW}{dx}\right|^2-a(\frac{d|W|^2}{dx})^2+a^2|W|^6\Big\}dx, \tag{1.2d}$$

$$F_6=\int_{-\infty}^{+\infty}\Big\{2\frac{d^3W}{dx^3}\frac{d^2\widetilde{W}}{dx^2}-5a\left|\frac{dW}{dx}\right|^2\frac{d|W|^2}{dx}-10a|W|^2\frac{d\widetilde{W}}{dx}\frac{d^2W}{dx^2}+5a^2|W|^4\widetilde{W}\frac{dW}{dx}\Big\}dx, \tag{1.2e}$$

where $\widetilde{W}$ represents the complex conjugation of W.

The NLSE above is an envelop wave equation [36] which appears in a variety of diverse physical systems, with successful applications to nonlinear optics, plasma physics and mechanics, depicting processes such as propagation of the electromagnetic field in optical fibers [16,26], the self-focusing and collapse of Langmuir waves [43], and the behavior of deep water waves in the ocean [6,28]. In the optical context, one can easily arrive at the NLSE from the Maxwell equations with nonlinear polarization when adopting envelop wave approximation by using multiscale techniques, which governs the time evolution of the slow amplitude of the wave packets [3,36]. Many research works have been done on the study of the NLSE in both the physical and the mathematical aspects of the equation. Some recent interests have been devoted to its external potentials, e.g., applications in Bose-Einstein condensates (BECs) [4,31]. The fast theoretical and experimental developments in nonlinear optics and condensed matter physics have drawn new attentions to the NLSE. An important emerging research topic is studying the discrete NLSE model (or spatial discretization version), e.g., an integrable type of discretization of the NLSE proposed by Ablowitz and Ladik [1,2] and accordingly referred to as the Ablowitz-Ladik (A-L) model:

$$i\frac{dW_l}{dt}+\frac{W_{l+1}-2W_l+W_{l-1}}{h^2}+\frac{a}{2}|W_l|^2(W_{l-1}+W_{l+1})=0, \tag{1.3}$$

and the most direct discrete NLSE is of the form

$$i\frac{dW_l}{dt}+\frac{W_{l+1}-2W_l+W_{l-1}}{h^2}+a|W_l|^2W_l=0, \tag{1.4}$$