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Spectral Elliptic Solvers in a Finite Cylinder

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Abstract. New direct spectral solvers for the 3D Helmholtz equation in a finite cylindrical region are presented. A purely variational (no collocation) formulation of the problem is adopted, based on Fourier series expansion of the angular dependence and Legendre polynomials for the axial dependence. A new Jacobi basis is proposed for the radial direction overcoming the main disadvantages of previously developed bases for the Dirichlet problem. Nonhomogeneous Dirichlet boundary conditions are enforced by a discrete lifting and the vector problem is solved by means of a classical uncoupling technique. In the considered formulation, boundary conditions on the axis of the cylindrical domain are never mentioned, by construction. The solution algorithms for the scalar equations are based on double diagonalization along the radial and axial directions. The spectral accuracy of the proposed algorithms is verified by numerical tests.

AMS subject classifications: 65N30, 65N35

Key words: Spectral elliptic solvers, Dirichlet and Neumann conditions, cylindrical coordinates, Legendre and Jacobi polynomials, uncoupled vector problem.

1 Introduction

Several problems in physics and engineering involve the solution of elliptic problems in cylindrical coordinates. Among the numerical techniques proposed so far are finite differences, spectral methods and spectral element methods. This work is concerned with spectral approximations. For a comprehensive theoretical framework see the monograph by Bernardi, Dauge and Maday [5].

The main difficulty with cylindrical coordinates lies in the axis singularity, the so-called 'pole' or 'centre problem'. However, the pole problem does not represent a true difficulty and can be actually turned into an opportunity when discretizing the equation. In fact, the Fourier components $u_m(r,z)$ of a scalar function $u(r,z,\phi)$ of the cylindrical

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coordinates (r,z,ϕ) must satisfy the following conditions to be a regular function of $\mathbb{R}^3$ [12]

$$u_m(r,z)=r^{|m|}U_m(r^2,z),$$

with function $U_m(s,z)$, $s\geq 0$, being a regular function of its two variables. If such conditions are not satisfied, the pole problem arises, since the numerical scheme provides an unwanted over-resolution near the cylinder axis which can severely limit the time step when time dependent problems have to be solved. On the other hand, the regularity conditions are helpful since they can be exploited to reduce the number of basis functions employed, by omitting the functions not satisfying them.

Several spectral methods have been proposed in the past in an attempt to satisfy the aforementioned regularity conditions [7]. In some cases, they are fulfilled only partially, as is the case, for instance, of shifted Chebyshev polynomials of quadratic argument, where the parity condition is satisfied, or Legendre-Galerkin, Chebyshev-Galerkin and Chebyshev-Legendre-Galerkin methods proposed by Shen and his co-workers [13,14,19], where just the essential conditions with the Helmholtz are satisfied. Bessel functions and polar Robert functions [16] satisfy all the conditions, but the former provide only algebraic convergence and the latter are severely ill conditioned. Other kinds of spectral approximations were proposed that disregarded completely the centre problem, such as tau-Chebyshev methods, see, e.g., [8] and Galerkin-Legendre collocation methods [6,17]. For instance, a direct spectral collocation method for the Poisson equation in polar and cylindrical coordinates has been proposed [9].

The first well conditioned basis that satisfies all the regularity conditions and provides spectral accuracy has been proposed by Matsushima and Marcus [15] in the context of the solution of 2D Neumann boundary value problems, see also Verkley [20]. Unfortunately, the condition numbers associated with Helmholtz operator grow as the fourth power of the degree for this basis. Moreover, its application to solve homogeneous and nonhomogeneous Dirichlet problems is unduly complicated.

This paper describes new direct spectral solvers for 3D scalar and vector elliptic equations supplemented by Dirichlet boundary conditions in a cylinder of finite axial extent. The solution algorithms are all based on a Galerkin variational formulation of the elliptic boundary value problem so that there is no need to introduce the idea of collocation. After the Fourier representation of the angular dependence, the spectral approximation of the unknown coefficients in the azimuthal plane is achieved by means of a new one-sided Jacobi polynomial basis for the radial variable and by bases of Legendre polynomials for the axial variable. A distinctive feature of the proposed 3D solvers is that the size of discrete subspaces associated with the radial operators is lower at higher Fourier wavenumbers.

In the case of the vector problem, the classical uncoupling of the radial and azimuthal components is employed. The uncoupling transformation is used in conjunction with the new Jacobi representation of variable size to implicitly satisfy all of the regularity conditions for the vector problem—a result which, to the best of our knowledge, is new.