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Fourth Order Schemes for Time-Harmonic Wave Equations with Discontinuous Coefficients

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Abstract. We consider high order methods for the one-dimensional Helmholtz equation and frequency-Maxwell system. We demand that the scheme be higher order even when the coefficients are discontinuous. We discuss the connection between schemes for the second-order scalar Helmholtz equation and the first-order system for the electromagnetic or acoustic applications.

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1 Introduction

We consider the one-dimensional linear Helmholtz equation:

$$\frac{d^2E}{dz^2}+k_0^2\nu(z)E=0, \quad z\in[0,Z_{\max}], \tag{1.1}$$

where the material coefficient $\nu(z)$ is assumed piecewise-constant. In this case, the solution $E(z)$ and its first derivative dE/dz are continuous everywhere [1], whereas the second and higher derivatives undergo jumps at the points of discontinuity of $\nu(z)$. A more complicated, nonlinear, version of Eq. (1.1) that arises in the context of nonlinear optics was analyzed and solved numerically in [1].

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Along with the second-order equation (1.1), we consider the first-order one-dimensional Maxwell equations in frequency space:

$$i\omega\epsilon E = \frac{dH}{dz}, \qquad i\omega\mu H = \frac{dE}{dz}, \tag{1.2}$$

where ϵ is piecewise constant and μ is constant. Eq. (1.1) with

$$k_0^2 \nu(z) = \omega^2 \epsilon\mu = \frac{\omega^2}{c^2}$$

can be easily obtained from system (1.2) by differentiating its second equation with respect to z and then substituting the derivative dH/dz from the first equation.

It has been recognized since the pioneering work of Kreiss and Oliger [7] that wave propagation equations require schemes with higher order accuracy due to phase errors and long time error accumulation. They found that the optimum scheme was between fourth- and sixth-order accurate. When the coefficients are only piecewise continuous it becomes much more difficult to construct higher order methods that retain their global accuracy. One approach to this difficulty has been the use of fictitious points as in the immersed interface and embedded boundary methods schemes first introduced by Zhang and LeVeque [15]. Later papers include [2, 8, 9, 16]. An analysis of the effect of discontinuous coefficients on the phase and amplitude errors was done by Gustafsson and Wahlund [4].

Our goal is to construct and test high order discrete approximations of (1.1) and (1.2) that keep the global higher order accuracy even in the presence of discontinuities in the coefficients. We will also examine connections between the resulting schemes similar to the previously identified relations [3] between a system and a scalar equation.

2 Fourth-order compact scheme for the Helmholtz equation

In this section we introduce the finite volume schemes for Eq. (1.1) based on its integral form. Let $a,b \in [0,Z_{\max}]$, $a<b$. We integrate (1.1) between the points a and b with respect to z:

$$\frac{dE(b)}{dz} - \frac{dE(a)}{dz} + k_0^2 \int_a^b \nu(z) E\, dz = 0. \tag{2.1}$$

Eq. (2.1) can be interpreted as the integral conservation law that corresponds to (1.1). For sufficiently smooth solutions, the two formulations are equivalent, see [1].

Following the approach in [1], we approximate the Helmholtz equation on a uniform grid with size h by applying the integral relation (2.1) between the midpoints of every two neighboring cells, i.e., for $[a,b] = [z_{m-\frac{1}{2}}, z_{m+\frac{1}{2}}]$, $m = 1,2,\ldots,M$. In addition, we assume that $\nu(z)$ may be discontinuous only at the grid nodes and denote by $\nu_{m+\frac{1}{2}}$ the value of ν