# A Discontinuous Galerkin Extension of the Vertex-Centered Edge-Based Finite Volume Method

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**Abstract.** The finite volume (FV) method is the dominating discretization technique for computational fluid dynamics (CFD), particularly in the case of compressible fluids. The discontinuous Galerkin (DG) method has emerged as a promising high-accuracy alternative. The standard DG method reduces to a cell-centered FV method at lowest order. However, many of today's CFD codes use a vertex-centered FV method in which the data structures are edge based. We develop a new DG method that reduces to the vertex-centered FV method at lowest order, and examine here the new scheme for scalar hyperbolic problems. Numerically, the method shows optimal-order accuracy for a smooth linear problem. By applying a basic $hp$-adaption strategy, the method successfully handles shocks. We also discuss how to extend the FV edge-based data structure to support the new scheme. In this way, it will in principle be possible to extend an existing code employing the vertex-centered and edge-based FV discretization to encompass higher accuracy through the new DG method.

## 1 Introduction

The finite volume (FV) method is currently the most widely used approach to discretize the equations of aerodynamics. The method balances exactly—with respect to the chosen numerical flux—the discrete values of mass, momentum, and energy between each control volume. The type of control volumes together with the choice of numerical flux

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determines which particular flavor of the FV method that is employed. Unstructured meshes are supported naturally by these methods, which allows for the treatment of flows around geometrically complex bodies.

The accuracy of FV methods is typically limited to first or second order. Efforts to increase the accuracy of the basic method include the so-called high-resolution schemes (MUSCL, ENO, WENO), which attain better flux approximations through extrapolation from directions where the solution is smooth. These schemes modestly increase the memory requirements, but the computational complexity grows with the order, since the improved accuracy relies on enlarging the width of the computational stencil. The regularity requirements on the mesh are thus likely to be high in order to obtain improved results.

A different approach to increase the accuracy is the discontinuous Galerkin (DG) method. It is a finite element method that does not explicitly enforce continuity between the elements as in a classic finite element method, but instead imposes a coupling between the solution at different elements with the use of numerical fluxes, as in the FV method. The DG method reduces to a FV method at lowest order, and can thus be viewed as a generalization of the FV method to higher orders. The increased order is not the result of an extrapolation procedure, as in the high-resolution schemes, but stems from a local approximation of the differential operator. The computational stencil of DG methods thus remains local regardless of order, and the quality of approximation can be expected not to depend as much on the mesh regularity as for the high-resolution FV schemes. On the other hand, the computational complexity and memory requirements increase sharply with order for the DG methods. Nevertheless, since a coarser mesh can be used, a higher-order DG method requires considerably less degrees of freedom to attain a solution with a given error bound, compared with a FV method.

There has been a strong development of the original DG method (Reed & Hill [24]) since the early nineties; Cockburn et al. [9] review the state of the art at the turn of the century. Hesthaven and Warburton give a thorough introduction to DG methods in their recent book [19]. Currently there is a coordinated effort in Europe, in which we participate, through the EU research project ADIGMA [3], which involves the development and assessment of different higher order methods such as DG and residual distribution schemes [2,10] for the next generation of CFD software aimed at the aeronautical industry.

Since DG is a generalization of the FV method, it is tempting to extend existing FV codes to encompass a DG method, in order to avoid a complete rewrite of large and sophisticated software systems. A serious hurdle for such a strategy is that the standard DG method is a higher-order version of the *cell-centered* FV method in which the control volumes coincide with the mesh cells (Fig. 1a), whereas many of today's codes are *vertex-centered* where the control volumes are constructed from a dual mesh, consisting in two dimensions of polygons surrounding each vertex in the original primal mesh (Fig. 1b). Some examples of vertex-centered FV codes are DLR-Tau [25], Edge [12], Eugenie [15], Fun3D [17], and Premo [26].