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A RESIDUAL A POSTERIORI ERROR ESTIMATOR FOR ELASTO-VISCOPLASTICITY

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Abstract. The numerical approximation of an elasto-viscoplastic problem is considered in this paper. Fully discrete approximations are obtained by using the finite element method to approximate the spatial variable and the forward Euler scheme to discretize time derivatives. We first recall an a priori estimate result from which the linear convergence of the algorithm is derived under suitable regularity conditions. Then, an a posteriori error analysis is provided. Upper and lower error bounds are obtained.

Key words. Elasto-viscoplasticity, fully discrete approximations, a posteriori error estimates, finite elements.

1. Introduction

Elasto-viscoplastic materials are very common in the real life because some types of rocks and metals can be modelled using a rate-type viscoplastic law. As noticed in [9], these materials allow both creep and relaxation phenomena.

In this work, we will consider a semilinear elasto-viscoplastic constitutive law introduced in [5] and already studied, from both mathematical and numerical point of views, by Ionescu and Sofonea (see the monograph [9] and the references cited therein). In particular, fully discrete approximations were considered in [6], where a priori estimates were obtained for an explicit Euler scheme. In this paper, this problem is revisited and a posteriori error analysis is performed in the study of that elasto-viscoplastic problem. This is done extending some arguments already applied in the study of the heat equation (see, e.g., [10, 11, 13]), some parabolic equations ([1]) or the Stokes equation ([2]). Recently, contact problems involving this kind of materials were studied (see the monograph [7] and the numerous references cited therein), and this work can be seen as a first step to deal with this interesting kind of contact problems (see [8] for an early study in the linear elasticity case).

The paper is structured as follows. In Section 2, the mechanical model and its variational formulation are described following the notation and assumptions introduced in [7]. Then, fully discrete approximations are provided in Section 3, by using the finite element method to approximate the spatial variable and the forward Euler scheme to discretize the time derivatives. In Section 4, an a priori error analysis obtained in [6] is recalled. Finally, using some results obtained in the study of the heat equation, an a posteriori error analysis is done in Section 5, providing an upper bound for the error, Theorem 5.1, and a lower bound, Theorem 5.2.

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2. Mechanical problem and its variational formulation

In this section, we present a brief description of the elasto-viscoplastic model (details can be found in [5, 9]).

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, denote a domain occupied by an elasto-viscoplastic body with a smooth boundary $\Gamma = \partial\Omega$ decomposed into two disjoint parts Γ_D and Γ_F such that meas $(\Gamma_D) > 0$. Moreover, let $[0, T]$, $T > 0$, be the time interval of interest and denote by $\boldsymbol{\nu}$ the unit outer normal vector to Γ.

Let $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$ be the spatial and time variables, respectively, and, in order to simplify the writing, we do not indicate the dependence of the functions on $\boldsymbol{x}$ and t. Moreover, a dot above a variable represents the derivative with respect to the time variable.

Let us denote by $\boldsymbol{u} = (u_i)_{i=1}^d$, $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1}^d$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{i,j=1}^d$ the displacement field, the stress tensor and the linearized strain tensor, respectively. We recall that

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The body is assumed elasto-viscoplastic and satisfying the following rate-type semi-linear constitutive law (see [5, 9]),

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})), \tag{1}$$

where $\mathcal{E}$ and $\mathcal{G}$ denote the fourth-order elastic tensor and the viscoplastic function, respectively.

We turn now to describe the boundary conditions.

On the boundary part Γ_D we assume that the body is clamped and thus the displacement field vanishes there (and so $\boldsymbol{u} = \boldsymbol{0}$ on $\Gamma_D \times (0, T)$). Moreover, we assume that a density of traction forces, denoted by $\boldsymbol{f}_F$, acts on the boundary part Γ_F; i.e.

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_F \quad \text{on} \quad \Gamma_F \times (0, T).$$

Denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ and by "$\cdot$" and $|\cdot|$ the inner product and the Euclidean norms on $\mathbb{R}^d$ and $\mathbb{S}^d$.

The mechanical problem of the quasistatic deformation of an elasto-viscoplastic body is then written as follows.

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times (0, T) \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times (0, T) \to \mathbb{S}^d$ *such that,*

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \textit{in} \quad \Omega \times (0, T), \tag{2}$$

$$-\mathrm{Div}\, \boldsymbol{\sigma} = \boldsymbol{f}_0 \quad \textit{in} \quad \Omega \times (0, T), \tag{3}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \textit{on} \quad \Gamma_D \times (0, T), \tag{4}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_F \quad \textit{on} \quad \Gamma_F \times (0, T), \tag{5}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \textit{in} \quad \Omega. \tag{6}$$

Here, $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ represent initial conditions for the displacement field and the stress tensor, respectively, and $\boldsymbol{f}_0$ denotes the density of body forces. Moreover, we notice that equilibrium equation (3) does not include the acceleration term because the problem is assumed quasistatic.

In order to obtain the variational formulation of Problem P, let $H = [L^2(\Omega)]^d$ and we define the following variational spaces:

$$V = \{\boldsymbol{w} \in [H^1(\Omega)]^d\,;\, \boldsymbol{w} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D\},$$
$$Q = \{\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^d \in [L^2(\Omega)]^{d\times d}\ ;\ \tau_{ij} = \tau_{ji},\ i, j = 1, \dots, d\}.$$

The following assumptions are required on the problem data.