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N-SIMPLEX CROUZEIX-RAVIART ELEMENT FOR THE SECOND-ORDER ELLIPTIC/EIGENVALUE PROBLEMS

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Abstract. We study the n-simplex nonconforming Crouzeix-Raviart element in approximating the n-dimensional second-order elliptic boundary value problems and the associated eigenvalue problems. By using the second Strang Lemma, optimal rate of convergence is established under the discrete energy norm. The error bound is also valid for the eigenfunction approximations. In addition, when eigenfunctions are singular, we prove that the Crouzeix-Raviart element approximates exact eigenvalues from below. Moreover, our numerical experiments demonstrate that the lower bound property is also valid for smooth eigenfunctions, although a theoretical justification is lacking.

Key words. n-simplex, nonconforming Crouzeix-Raviart element, second order elliptic equation, error estimates, eigenvalues, lower bound.

1. Introduction

Nonconforming finite elements have attracted much attention in scientific computing community. In some recent works, Morley element, Adini element, Bogner-Fox-Schmit element, and Zienkiewicz-type element have been extended into arbitrary dimensions by Wang, Shi, and Xu [13, 14]. In this paper, we study the n-simplex nonconforming Crouzeix-Raviart element.

The triangular Crouzeix-Raviart element was first introduced in 1973 [6] to solve the stationary Stokes equation. This element was also used to solve the second-order elliptic problems [12] and linear elasticity equations [3, 7]. Recently, Armentano and Durán proved that the triangular Crouzeix-Raviart element approximates the eigenvalue of the Laplace operator from below under certain conditions [1]. All above mentioned works are in the two dimensional setting. Indeed, the Crouzeix-Raviart element has its n-dimensional extension [5]. We shall apply it to solve higher-dimensional second-order elliptic equations here. With help of the second Strang lemma, we establish the optimal rate of convergence in the discrete energy norm. This result is then extended to eigenfunctions of the associated eigenvalue problems. We prove that when the eigenfunction is singular, the numerical eigenvalue obtained by the Crouzeix-Raviart element approximates the exact one from below. This theoretical result is illustrated by numerical examples. Moreover, our numerical experiments indicate that the lower bound property is also valid for smooth eigenfunctions, at least for the Laplace operator on the cube.

By the min-max principle, a conforming finite element results in an upper bound for eigenvalue problems associated with second-order elliptic operators. The fact that the nonconforming Crouzeix-Raviart element provides a lower bound has a significant impact from the *a posteriori* error control view point. By comparing the two, we are able to control the error by a given tolerance. That is why the

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subject of non-conforming elements approximating exact eigenvalues from below has attracted much attention in scientific community. Other than [1] mentioned above, Rannacher [11] gave numerical examples about Morley element and Adini element in approximating the exact eigenvalues from below for a plate vibration problem; Yang [15] proved that Adini element approximates the exact eigenvalues from below for the plate vibration problem; Lin and Lin [9] proved that the non-conforming EQ_1^{rot} approximates the exact eigenvalues of the Laplace operator from below; Zhang, Yang and Chen [17] proved that the non-conforming Wilson element approximates the exact eigenvalues of the Laplace operator from below. Again, all above works are for the two dimensions. The results in this paper are for any n-dimension.

2. Approximation of second-order elliptic problems

Consider the second-order elliptic boundary value problem on a polygonal domain $\Omega \subset R^n$,

$$Lu \equiv -\sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u) + au = f, \quad \text{in } \Omega; \qquad u = 0, \quad \text{on } \partial\Omega. \tag{2.1}$$

We assume that $a_{ij} = a_{ji}, a_{ij} \in W_{1,\infty}(\Omega)$, $a \in L_\infty(\Omega)$, $a \geq 0$, $f \in L_2(\Omega)$, and there exists a constant $\beta > 0$, such that $\sum\limits_{i,j=1}^{n} a_{ij}\xi_j\xi_i \geq \beta \sum\limits_{i=1}^{n} \xi_i^2$ a.e. in Ω for all $(\xi_1, \xi_2, \cdots, \xi_n) \in R^n$.

The weak form of (2.1) is to seek $u \in H_0^1(\Omega)$ such that

$$a(u,v) = b(f,v), \quad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where,

$$a(u,v) = \int_\Omega \{\sum_{i,j=1}^{n} a_{ij}\partial_j u \partial_i v + auv\}dx, \quad b(f,v) = \int_\Omega fv dx, \quad \|u\|_b = \|u\|_{0,2}.$$

Then the bilinear form $a(\cdot,\cdot)$ is $H_0^1(\Omega)$-elliptic, continuous, and symmetric over the product space $H_0^1(\Omega) \times H_0^1(\Omega)$.

In the sequel, we need the following *a prior* estimate:

$$\|u\|_{2,p} \leq C(p)\|f\|_{0,p}, \quad p \in (1,\infty). \tag{2.3}$$

Remark. It is well known that (2.3) is valid when $\partial\Omega$ is $C^{1,1}$. When Ω is an n-cube, (2.3) is valid for $c(p) = \max(p, p/(p-1))$, see [4, Theorem 2.3.4], for the proof.

Let π_h be an n-simplex partition for Ω, and let the barycenters of the $n+1$-faces of an n-simplex be $z_1, z_2, z_3, \cdots, z_{n+1}$. Then the non-conforming Crouzeix-Raviart finite element space is,

$S^h = \{v \in L_2(\Omega) : v\,|_K \in P_1(K),\ \forall K \in \pi_h,\ v$ is continuous at z_j, and $v = 0$ at barycenters on $\partial\Omega\}$. Clearly, $S^h \not\subset H_0^1(\Omega)$.

In this paper, we suppose that the family of triangulations π_h is regular (see [5, P131]).

The non-conforming Crouzeix-Raviart finite element approximation of (2.1) is to seek $u_h \in S^h$ such that

$$a_h(u_h, v) = b(f,v),\ \forall v \in S^h, \tag{2.4}$$