English Series | NUMERICAL MATHEMATICS A Journal of Chinese Universities | Vol.14, No.4 Nov. 2005

NEW ERROR EXPANSION FOR ONE-DIMENSIONAL FINITE ELEMENTS AND ULTRACONVERGENCE*

Chen Chuanmiao(陈传淼) Xie Ziqing(谢资清) Liu Jinghong(刘经洪)

Abstract *Based on an improved orthogonal expansion in an element, a new error expression of n-degree finite element approximation u_h to two-point boundary value problem is derived, and then superconvergence of two order for both function and derivatives are obtained.*

Key words *finite element, one-dimension, new error expansion, ultraconvergence.*
AMS(2000)subject classifications 65N30

1 Introduction

Consider two-point boundary value problem

$$Lu = -(a(x)u')' - (b(x)u)' + c(x)u = f \ \text{ in } \Omega = (0,1),\ u(0) = u'(1) = 0, \tag{1}$$

where the coefficients and f are suitably smooth, $a(x) \geq a_0 > 0$. The corresponding weak formulation is to find $u \in S_0 = \{u \in H^1(\Omega), u(0) = 0\}$ such that

$$A(u,v) = (f,v),\ v \in S_0,\ A(u,v) = \int_0^1 (au'v' - (bu)'v + cuv)\mathrm{d}x. \tag{2}$$

Below always assume that the bilinear form $A(u,v)$ is bounded and S_0-coercive. Denote by $W^{k,p}(\Omega)$ Sobolev space with norm $||u||_{k,p,\Omega}$. If $p = 2$, simply use $H^k = W^{k,2}$ and $||u||_{k,\Omega}$.

Denote a uniform partition of Ω by $Z_h:\ 0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$. Set an element $\tau_j = (x_{j-1}, x_j)$, its midpoint $\bar{x}_j = (x_{j-1} + x_j)/2$ and half steplength $h_j = (x_j - x_{j-1})/2$. Denote piecewise n-degree finite element subspace by

$$S_0^h = \{v \in C(\bar{\Omega}), v|_{\tau_j} \in P_n, j = 1, 2, \cdots, N, v(0) = 0\}.$$

Define the finite element solution $u_h \in S_0^h$ of (2) such that

$$A(u_h, v) = (f, v), \quad v \in S_0^h. \tag{3}$$

* This work was supported by The Special Funds for State Major Basic Research Projects (No. G1999032804) and The National Natural Science Foundation of China (Grant No.10471038).
Received: Jun. 2, 2002.

From (2) and (3), the error $e = u - u_h$ satisfies the following orthogonality relation

$$A(e, v) = 0, \quad v \in S_0^h. \tag{4}$$

The idea of Element Orthogonality Analysis (EOA) is to construct a n degree approximation $u_I \in S_0^h$ which is superclose to u_h. From (4) we see that it is equivalent to that the remainder $R = u - u_I$ satisfies ([12],[2])

$$\begin{aligned} A(u_h - u_I, v) = A(R, v) &= O(h^{n+\alpha})||v||_{1,1,\Omega}, \ \alpha > 0, \\ \text{or } &= O(h^{n+1+\alpha})||v||^*_{1,2,\Omega}, \ \alpha > 0, \end{aligned} \tag{5}$$

where $||v||^*_{2,1,\Omega} = \sum_\tau ||v||_{2,1,\tau}$ is the mesh norm. By the help of discrete Green function $g_h \in S_0^h$, $||g_h||^*_{2,1,I} \le C$ (or the gradient type Green function $G_h \in S_0^h, ||G_h||_{1,1,I} \le C$), we can derive superconvergence estimates

$$u_h - u_I = O(h^{n+1+\alpha}), \text{ or } D_x(u_h - u_I) = O(h^{n+\alpha}).$$

From the equalities

$$u - u_h = R + O(h^{n+1+\alpha}), D_x(u - u_h) = DR + O(h^{n+\alpha}),$$

we know that the roots of remainder R (or DR) are superconvergence points of u_h (or Du_h). Therefore, the key of EOA is first how to think out the desired superclose function u_I. Fortune is that based on some orthogonal expansion in an element, we have found this kind of superclose function and got many basic seperconvergence results (See [2] for one-dimension elements and [3] for rectangular elements and so on). This paper will further improve the orthogonal expansion, derive more exact error expression, and then get sharp superconvergence results. The importance of the improvement consists in that which can also be applied to high order equations and multidimensional cases. The relative results will be discussed in other papers later.

Take the transformation $x = ht, t \in T = (-1, 1)$ in a standard element $\tau = (-h, h)$. Set still $u(t) = u(ht)$. Obviously $\partial_t^i u = D_x^i u(x) h^i$. Introduce Legendre polynomials in T

$$l_0 = 1, l_1 = t, l_2 = \frac{1}{2}(3t^2 - 1), l_3 = \frac{1}{2}(5t^3 - 3t), \cdots, l_n = \frac{1}{2^n n!}\partial_t^n (t^2 - 1)^n, \tag{6}$$

where the inner product $(l_i, l_j) = 0$ if $i \ne j$, otherwise $(l_j, l_j) = \dfrac{2}{2j+1}$. $l_n(t)$ has n distinct roots (n order Gauss points) $t_1' < t_2' < \cdots < t_n'$ in T. Integrating l_n, we get another M-type family of polynomials [2]

$$M_0 = 1, M_1 = t, M_3 = \frac{1}{2}(t^2 - 1), M_3 = \frac{1}{2}(t^3 - t), \cdots, M_{n+1} = \frac{1}{2^n n!}\partial_t^{n-1}(t^2 - 1)^n, \tag{7}$$

which has the following quasi-orthogonal property: $(M_i, M_j) \ne 0$ if $i - j = 0$ or $= \pm 2$, otherwise $(M_i, M_j) = 0$. Obviously $M_j(\pm 1) = 0$ for $j \ge 2$. $M_n(t)$ has n distinct roots (n order Lobatto points) $t_0 = -1 < t_1 < t_2 < \cdots < t_n = 1$.

Any smooth function $u(t)$ in T can be expanded as an orthogonal series

$$u(t) = \sum_{j=0}^{\infty} \beta_j l_j(t), \quad \beta_j = \gamma_j(u, l_j), \ \gamma_j = j + 1/2. \tag{8}$$