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An Artificial Boundary Condition for the Vortex Movements in Two Dimensions

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Abstract. An approximate artificial boundary condition based on a boundary integral equation is designed for the vortex movements. Point vortex and cloud in cell methods are used in numerical simulation of vortex motions. The numerical experiments show that the approximate artificial boundary condition is useful and sufficiently accurate in hydrodynamics.

Key words: Artificial boundary condition; boundary integral equation; point vortex method; cloud in cell method.

AMS subject classifications: 65D07, 65D32, 65F10, 65L10, 65N22

1 Motion of vortices and cloud in cell method

The motion of incompressible inviscid flow in two dimensions can be described by the equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla) + \frac{1}{\rho}\nabla P = \mathbf{f} \tag{1}$$

$$\nabla \mathbf{u} = 0 \ , \tag{2}$$

where $\mathbf{u} = (u, v), \rho, P$, and $\mathbf{f} = (f_1, f_2)$ denote fluid velocity, density, pressure, and force acting on a unit fluid at the point $\mathbf{x} = (x, y)$ and time t, respectively. We introduce the vorticity distribution $\omega(x, y, t) = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ and the stream function $\psi(x, y, t) = \int_{\mathbf{x_0}}^{\mathbf{x}} vdx + udy$, where $\mathbf{x_0} = (x_0, y_0)$ is an arbitrary point in the $(x, y)-$plane. The above integration is independent of the integration path. Due to equations (2) and Green's formula, the equations (1), (2) then become

$$\begin{aligned}
\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} &= 0 && (3)\\
-\Delta\psi &= \omega && (4)\\
\frac{\partial \psi}{\partial y} &= u && (5)\\
\frac{\partial \psi}{\partial x} &= -v && (6)\\
\omega(x, y, 0) &= \omega_0(x, y) \ . && (7)
\end{aligned}$$

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In order to simulate numerically the motion of vortices, we adopt the point vortex method (PVM) [1,2]. It starts from discretising the continuous vorticity distribution ω by a large number of point vortices, then simulating the motion of vortices by tracing every point vortex. By covering the compact support of ω with a rectangular grid with mesh spacing h_x, h_y in the x, y directions respectively, we get M rectangles. The k-th point vortex, with vorticity $p_k = \omega(x_k, y_k, t)h_x h_y$, is located at the center (x_k, y_k) of the k-th rectangle, and satisfies the motion equations

$$\frac{\partial x_k}{\partial t} = u_k, \quad \frac{\partial y_k}{\partial t} = v_k. \tag{8}$$

Then the vorticity distribution is approximated by the formula

$$\omega(x, y, t) \approx \sum_{k=1}^{M} p_k \delta(x - x_k)\delta(y - y_k),$$

and

$$\sum_{k=1}^{M} p_k = \iint_{\Omega} \omega(x, y, t) d\sigma.$$

Now we divide the computational domain into uniform squares with mesh spacing h in each the direction, and use the cloud in cell method (CICM) [1] to compute the vorticity of the mesh points. If the coordinates of a point vortex p_k are written as $x_k = ih + dx, y_k = jh + dy$, the point vortex allocates its vorticity p_k to the four surrounding mesh points by CICM:

$$\begin{aligned} \omega_{i,j} &= \frac{(h - dx)(h - dy)}{h^2} p_k, & \omega_{i+1,j} &= \frac{dx(h - dy)}{h^2} p_k, \\ \omega_{i+1,j+1} &= \frac{dxdy}{h^2} p_k, & \omega_{i,j+1} &= \frac{(h - dx)dy}{h^2} p_k. \end{aligned} \tag{9}$$

When all point vortices have credited their vortices to the mesh points in the same way, $Poisson's$ equation (4) can be solved by the usual five-point scheme with an artificial boundary condition. In this paper we will adopt an artificial condition based on a boundary integral equation. When the stream function ψ is obtained, the velocity of the mesh points can be obtained by center differences approximating the equations (5) and (6). The velocity of the point vortex p_k is evaluated by the CICM according to

$$\begin{aligned} u_k &= \frac{(h - dx)(h - dy)}{h^2} u_{i,j} + \frac{dx(h - dy)}{h^2} u_{i+1,j} + \frac{dxdy}{h^2} u_{i+1,j+1} + \frac{(h - dx)dy}{h^2} u_{i,j+1}, \\ v_k &= \frac{(h - dx)(h - dy)}{h^2} v_{i,j} + \frac{dx(h - dy)}{h^2} v_{i+1,j} + \frac{dxdy}{h^2} v_{i+1,j+1} + \frac{(h - dx)dy}{h^2} v_{i,j+1}. \end{aligned} \tag{10}$$

Using the differences scheme

$$x_k(t + \Delta t) = x_k(t) + u_k \Delta t\ , \quad y_k(t + \Delta t) = y_k(t) + v_k \Delta t, \tag{11}$$

where Δt is the time step, we advance the point vortices one time step.