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CONVERGENCE OF THE CYCLIC REDUCTION ALGORITHM FOR A CLASS OF WEAKLY OVERDAMPED QUADRATICS*

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Abstract

In this paper, we establish a convergence result of the cyclic reduction (CR) algorithm for a class of weakly overdamped quadratic matrix polynomials without assumption that the partial multiplicities of the nth largest eigenvalue are all equal to 2. Our result can be regarded as a complement of that by Guo, Higham and Tisseur [SIAM J. Matrix Anal. Appl., 30 (2009), pp. 1593-1613]. The numerical example indicates that the convergence behavior of the CR algorithm is largely dictated by our theory.

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1. Introduction

The quadratic eigenvalue problems (QEPs) are to find scalars λ and nonzero vectors x and y satisfying $Q(\lambda)x = 0$ and $y^*Q(\lambda) = 0$, where

$$Q(\lambda) = \lambda^2 A + \lambda B + C \qquad \text{with} \quad A, B, C \in \mathbb{C}^{n\times n} \tag{1.1}$$

is a quadratic matrix polynomial (or the quadratic for the brevity). Vectors x and y are right and left eigenvectors corresponding to the eigenvalue λ. QEPs have extensive applications in practical engineering problems. We refer to [18] for a good review.

In this paper, we consider the overdamped QEP which belongs to a class of hyperbolic QEPs with the following definition [6].

Definition 1.1. *The quadratic $Q(\lambda)$ is called hyperbolic if A, B, C are all Hermitian, A is positive definite, and*

$$(x^*Bx)^2 > 4(x^*Ax)(x^*Cx) \quad for \quad all \quad nonzero \quad x \in \mathbb{C}^n.$$

A hyperbolic QEP can be transformed into an overdamped one [11], so there is no loss of generality to consider only overdamped problems.

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Definition 1.2. *The quadratic $Q(\lambda)$ is called overdamped if it is hyperbolic with positive definite B and positive semidefinite C.*

It is known that an overdamped quadratic has $2n$ real, nonpositive and semisimple eigenvalues that can be ordered $0 \geq \lambda_1 \geq \cdots \geq \lambda_n > \lambda_{n+1} \geq \cdots \geq \lambda_{2n}$ [18]. When the nth largest and the nth smallest eigenvalues coalesce (i.e. $\lambda_n = \lambda_{n+1}$), the quadratic is called *weakly overdamped* (WO) in the terminology of Markus [16] (see also [8, Sec. 5]). The following lemma collects some properties of a weakly overdamped quadratic [6, 16].

Lemma 1.1. *Let $Q(\lambda)$ be a WO quadratic.*

(a) $Q(\lambda)$ has $2n$ real eigenvalues that can be ordered $0 \geq \lambda_1 \geq \cdots \geq \lambda_n = \lambda_{n+1} \geq \cdots \geq \lambda_{2n}$. The partial multiplicities[1] of λ_n are at most 2, and the eigenvalues other than λ_n are semisimple.

(b) The associated quadratic matrix equation (QME)

$$Q(S) = AS^2 + BS + C = 0 \tag{1.2}$$

admits two extremal solutions (also called solvents in the matrix polynomial theory) $S^{(1)}$ (the primary) and $S^{(2)}$ (the secondary), whose eigenvalues are the n largest and the n smallest roots of $Q(\lambda)$, respectively.

Recently, Guo, Higham and Tisseur [6] devised an efficient algorithm to detect and solve the overdamped QEPs. This algorithm was based on the cyclic reduction (CR) (stated in Section 3) with quadratic convergence. They also showed that, for WO QEPs, the convergence of the CR algorithm became linear with a constant at worst 1/2. We note that their analysis in that case needs the requirement that the partial multiplicities of the nth largest eigenvalue (i.e. λ_n) are all equal to 2. However, we know from Lemma 1.1 that the partial multiplicities of λ_n are at most 2. So one may wonder: (i) Are there any WO quadratics with the partial multiplicities of λ_n containing both 1 and 2? (ii) If such WO quadratics exist, what is the behavior of the CR algorithm for them?

The purpose of this paper is to investigate the above two issues. We give an example in the Section 3 to show that it does exist the WO quadratic with the partial multiplicities of λ_n containing both 1 and 2. However, the convergence behavior of the CR algorithm for such a quadratic is some different with that in [6]. We also try to establish a convergence of the CR algorithm for general WO quadratics (with no assumption on the partial multiplicities of λ_n). Unfortunately, the attempt seems not easy since the structure of the corresponding eigenspace is indefinite. So we instead construct a canonical diagonal quadratic in which the partial multiplicities of λ_n include 1 and 2. Then we extend the diagonal quadratic to a class of (not all) isospectral[2] WO quadratics. Since the structure of eigenspace for such extended quadratics can be made out, we can obtain the convergence theorem of the CR algorithm by another equivalent doubling algorithm (see [2, 6, 15, 22]). The derived theorem (unlike that in [6]) indicates that some matrix sequences generated by the CR algorithm no longer converge to the zero matrix if the partial multiplicities of λ_n contain 1. Therefore, our result can be seen as a complement of convergence for the CR algorithm.

[1] The partial multiplicities of an eigenvalue of $Q(\lambda)$ are the sizes of the Jordan blocks in which it appears in a Jordan matrix of $Q(\lambda)$ [5].

[2] The term "isospectral" is in the sense that the eigenvalues and all their partial multiplicities are common to isospectral matrix polynomial [13].