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A SET OF SYMMETRIC QUADRATURE RULES ON TRIANGLES AND TETRAHEDRA*

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Abstract

We present a program for computing symmetric quadrature rules on triangles and tetrahedra. A set of rules are obtained by using this program. Quadrature rules up to order 21 on triangles and up to order 14 on tetrahedra have been obtained which are useful for use in finite element computations. All rules presented here have positive weights with points lying within the integration domain.

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1. Introduction

In finite element computations, numerical integration is widely used for computing integrals of functions or bilinear forms. For triangular meshes numerical integrations on line segments, triangles, and tetrahedra are needed. In contrast to quadrilaterals or hexahedra on which quadrature formulas can be naturally derived from tensor products of one-dimensional Gauss quadrature rules, high-order non-tensor product quadrature rules on triangles and tetrahedra are difficult to construct. In fact, many of the non-tensor product rules published in finite element textbooks contain either negative weights or points outside of the integration domain, which are undesirable for numerical computations. As a result, it is a common practice in many finite element packages to use quadrature rules associated with tensor products of one of the Gauss-Jacobi rules; these rules are unsymmetric and generally require (as many as twice in three-dimensions) more function evaluations.

There have been many studies searching for quadrature rules on triangles and tetrahedra, both numerically and analytically. The problem of finding quadrature rules generally leads to a problem of finding the zeros or minima of high-order multi-variate polynomials, which is known to be extremely difficult. Many methods have been developed for computing quadrature rules; we refer the reader to [1, 2, 3, 4, 5, 6, 7, 8, 9] and the references therein.

In this paper, we present a program for computing symmetric quadrature rules on triangles and tetrahedra and a set of quadrature rules obtained using this program. The underlying algorithm turns the problem of computing quadrature rules into nonlinear least square solution of systems of polynomial equations, and makes use of MINPACK [10] which is a publicly available well known minimization package. All rules presented here are fully symmetric and have positive weights with quadrature points lying within the integration domain. We believe that at least some of the rules presented in this paper are new. We prefer symmetric rules

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on triangles and tetrahedra because they are naturally related to the geometric symmetry of the integration domain and can be represented in a compact form using symmetry orbits, and more importantly, the symmetry of the quadrature points may be exploited, together with the symmetry of finite element basis functions, to reduce the computational cost in the calculation of mass and stiffness matrices.

After this short introduction, the rest of the paper is organized as follows. In Section 2 we give a brief description on quadrature rules and define some notations used. In Section 3 we present our program and the underlying numerical algorithm for computing quadrature rules. In Section 4 we report the quadrature rules found using this program. In the final section we give some concluding remarks.

2. Notations

Let T be a d-dimensional simplex, here $d = 2$ (triangle) or 3 (tetrahedron). A quadrature rule $\mathcal{R}$ on T is defined as a set of point and weight pairs: $\mathcal{R} = \{(p_i, w_i) \mid i = 1, \cdots, n\}$, such that for any function $f(x)$ defined on a domain containing T and the points p_i, its integral on T can be approximated by:

$$\int_T f(x)dx \approx |T| \sum_{i=1}^{n} f(p_i)w_i, \tag{2.1}$$

where $n \in \mathbb{N}$ is the number of points, p_i are the quadrature points, w_i are the associated weights, $|T|$ denotes the area ($d = 2$) or volume ($d = 3$) of T.

A quadrature rule is said to be *of (algebraic) order p* if (2.1) is exact for all polynomials of degree not exceeding p. It is clear that if a quadrature rule is of order 0, then the sum of the weights must be equal to 1.

When dealing with a simplex it is often convenient to use barycentric coordinates. Let v_i, $i = 1, \cdots, d+1$, be the vertices of T. Then the barycentric coordinates $(\xi_1, \cdots, \xi_{d+1})$ of a point p with respect to T is determined by:

$$p = \sum_{i=1}^{d+1} \xi_i v_i \quad \text{and} \quad \sum_{i=1}^{d+1} \xi_i = 1.$$

Barycentric coordinates are invariant under affine transformations and $p \in T$ if and only if all its barycentric coordinates lie in the interval $(0, 1)$.

A quadrature rule $\mathcal{R}$ is said to be *symmetric* if it is invariant under permutations of the barycentric coordinates. More precisely, let $(\xi_1, \cdots, \xi_{d+1})$ be a quadrature point of $\mathcal{R}$ associated with weight w, then for any permutation $i_1, \cdots, i_{d+1}$ of the indices $1, \cdots, d+1$, the point $(\xi_{i_1}, \cdots, \xi_{i_{d+1}})$ is also a quadrature point of $\mathcal{R}$ associated with the same weight. For a symmetric quadrature rule, the set of quadrature points can be naturally divided into symmetry orbits, with each orbit containing all the points generated by permuting the barycentric coordinates of a single point. The symmetry orbits can be classified into a number of permutation stars, which are summarized in Tables 2.1 and 2.2, in which the notations for the permutation stars are from [11] which have the advantage of easily distinguishing stars on triangles and on tetrahedra.

3. The Numerical Algorithm

Denote by $\mathbb{P}^{(p)}$ the set of polynomials of degree less than or equal to p. The problem of finding an n-point quadrature rule of order p consists of finding the quadrature points p_i and