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THREE WAY DECOMPOSITION FOR THE BOLTZMANN EQUATION*

Ilgis Ibragimov
Elegant Mathematics Ltd., Hanauer Muehle 2, 66564 Ottweiler, Germany
Email: ii@elegant-mathematics.com
Sergej Rjasanow
Department of Mathematics and Computer Science, University of Saarland, Saarbrücken, Germany
Email: rjasanow@num.uni-sb.de

Abstract

The initial value problem for the spatially homogeneous Boltzmann equation is considered. A deterministic numerical scheme for this problem is developed by the use of the three way decomposition of the unknown function as well as of the collision integral. On this way, almost linear complexity of the algorithm is achieved. Some numerical examples are presented.

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1. Introduction

The object of our considerations is the initial value problem for the classical spatially homogeneous Boltzmann equation

$$\frac{\partial}{\partial t} f(t,v) = Q(f,f)(t,v), \quad t \in \mathbb{R}_+, \quad f(0,v) = f_0(v), \quad v \in \mathbb{R}^3, \tag{1.1}$$

which describes the time evolution of the particle density

$$f : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}_+$$

from its initial value f_0 to the final Maxwell distribution

$$\lim_{t\to\infty} f(t,v) = f_M(v) = \frac{\varrho_0}{(2\pi\, T_0)^{3/2}}\, e^{-\frac{|v-V_0|^2}{2\,T_0}}\ . \tag{1.2}$$

The right-hand side of the equation (1.1), known as the collision integral or the collision term, is of the form

$$Q(f,f)(t,v) = \int\limits_{\mathbb{R}^3}\int\limits_{S^2} B(v,w,e)\big(f(t,v')f(t,w') - f(t,v)f(t,w)\big)\, de\, dw. \tag{1.3}$$

The following notations have been used in (1.3): $v, w \in \mathbb{R}^3$ are the pre-collision velocities, $e \in S^2 \subset \mathbb{R}^3$ is a unit vector, $v', w' \in \mathbb{R}^3$ are the post-collision velocities, and $B(v,w,e)$ is the collision kernel. The operator $Q(f,f)$ represents the change of the distribution function f due to the binary collisions between particles. A single collision results in the change of the velocities of the colliding partners

$$v, w \to v', w'. \tag{1.4}$$

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The collision transformation (1.4) conserves the momentum and the energy

$$v+w=v'+w', \quad |v|^2+|w|^2=|v'|^2+|w'|^2.$$

It can be written in the following form

$$v'=\frac{1}{2}\Big(v+w+|u|e\Big), \quad w'=\frac{1}{2}\Big(v+w-|u|e\Big), \quad e\in S^2,$$

where $u=v-w$ denotes the relative velocity of the colliding particles. We will deal with an isotropic cut-off kernel B, namely with the Variable Hard Spheres model (VHS) [3]

$$B(v,w,e)=C_\lambda\,|u|^\lambda, \quad -3<\lambda\le 1. \tag{1.5}$$

The model includes, as particular cases, the hard spheres model for $\lambda=1$ and a special case of the Maxwell pseudo–molecules with $\lambda=0$.

All relevant physical values of the gas flow are computed as the first 13 moments of the distribution function or their combinations. These moments are: the density

$$\varrho(t)=\int\limits_{\mathbb{R}^3} f(t,v)\,dv, \tag{1.6}$$

the momentum

$$m(t)=\int\limits_{\mathbb{R}^3} v\,f(t,v)\,dv, \tag{1.7}$$

the momentum flow

$$M(t)=\int\limits_{\mathbb{R}^3} vv^\top f(t,v)\,dv, \tag{1.8}$$

and the energy flow

$$r(t)=\frac{1}{2}\int\limits_{\mathbb{R}^3} v|v|^2 f(t,v)\,dv. \tag{1.9}$$

Note that the matrix M is symmetric and therefore defined by its upper triangle. Using these moments, we define the bulk velocity

$$V(t)=m(t)/\varrho(t), \tag{1.10}$$

the internal energy and the temperature

$$e(t)=\frac{1}{2\,\varrho(t)}\big(\mathrm{tr}M(t)-\varrho(t)|V(t)|^2\big)\,, \quad T(t)=\frac{2}{3}e(t), \tag{1.11}$$

the pressure

$$p(t)=\varrho(t)T(t), \tag{1.12}$$

the stress tensor

$$P(t)=M(t)-\varrho(t)V(t)V(t)^\top,$$

and the heat flux vector

$$q(t)=r(t)-\left(M(t)+\Big(\frac{1}{2}\mathrm{tr}M(t)-\varrho(t)|V(t)|^2\Big)I\right)V(t).$$

Note that in the spatially homogeneous case we consider here, the following important conservation properties hold. The density, the momentum, and the trace of the momentum flow