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EXPLICIT ERROR ESTIMATES FOR MIXED AND NONCONFORMING FINITE ELEMENTS*

Shipeng Mao and Zhong-ci Shi
LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
Email: maosp@lsec.cc.ac.cn shi@lsec.cc.ac.cn

Abstract

In this paper, we study the explicit expressions of the constants in the error estimates of the lowest order mixed and nonconforming finite element methods. We start with an explicit relation between the error constant of the lowest order Raviart-Thomas interpolation error and the geometric characters of the triangle. This gives an explicit error constant of the lowest order mixed finite element method. Furthermore, similar results can be extended to the nonconforming P_1 scheme based on its close connection with the lowest order Raviart-Thomas method. Meanwhile, such explicit a priori error estimates can be used as computable error bounds, which are also consistent with the maximal angle condition for the optimal error estimates of mixed and nonconforming finite element methods.

Mathematics subject classification: 65N12, 65N15, 65N30, 65N50.
Key words: Mixed finite element, Nonconforming finite element, Explicit error estimate, Maximal angle condition.

1. Introduction

Finite element methods for the accurate numerical solution of partial differential equations are of great practical interest in the engineering and scientific computing applications. Up to now, their mathematical theory such as a priori error estimates have been well established in the literature, see, e.g., [9, 14, 36]. Let u, u_h denote the exact solution of the model problem and the associated discretized solution, respectively. The convergence analysis of finite element method is typically of the form

$$\|u - u_h\| \le Ch^k|u|, \tag{1.1}$$

where h denotes the maximal diameter of the triangulation, $\|\cdot\|$ and $|\cdot|$ stand for some appropriate norm and seminorm in certain function spaces, respectively.

Such a result may not be effective unless the dependence of the constant C is specified. The classical finite element theories, see, e.g., [9, 14], show that the constant C in (1.1) does not dependent on the function u, but may dependent on the sine of the minimal angle of the triangulation for the two dimensional case, which is equivalent to the well-known nondegenerate assumption or regular assumption of finite element meshes. In fact, the minimal angle condition for the finite elements can be relaxed, which results in the so-called degenerate elements. Error estimates for degenerate elements can go back to the works by Babuška and Aziz [5] and by Jamet [20]; both of them proved the optimal error estimate for the linear Lagrange triangular element under the assumption that the underlying meshes satisfy the maximal angle condition.

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Since late 1980's degenerate elements have been extensively studied; interested readers are referred to [2, 12, 22] and references therein.

As is known that there appear various constants in the process to derive the error estimates. It is good to evaluate these constants explicitly for a quantitative error bound purpose. Actually, there are some works on an explicit error estimate of the finite element methods, see, e.g., [3, 6, 7, 18, 21] for linear finite element methods and [25] for bilinear quadrilateral finite element methods. However, almost all of them are concentrated on the standard conforming finite element methods, which only involves an explicit interpolation error estimate. To the best of our knowledge, as far as other type finite element methods are concerned, for example, mixed elements and nonconforming elements, there seem no explicit error bounds are given. In order to obtain an explicit error bounds for such type elements, only having the interpolation error estimate is not enough. The mixed element methods and the nonconforming element methods need further an explicit bound of the discrete inf-sup constant and of the consistency error, respectively.

In this paper, we are aim to obtain an explicit error bound for the lowest order mixed finite element and nonconforming finite element for the second order problems ([33, 34]). Firstly, we prove some results on the error constants of the Raviart-Thomas interpolation, which plays an essential role in the a priori error estimates of finite element methods. The technical tool is an explicit trace theorem on the reference unit triangle. On the other hand, the Babuška-Brezzi condition is well-known to guarantee the stability of a mixed finite element and play a key role in the error estimates (cf. [10, 11]). It is also essential to give an explicit expression of the inf-sup constant. Based on these results we can derive a constructive error bound for the mixed finite element. Finally, we also obtain an explicit error estimate for the nonconforming Crouzeix-Raviart [16] element by its close relation to the mixed finite element method (cf. [4, 26]). Note that Kikuchi and Liu [21] recently derived an explicit interpolation error bounds for the nonconforming Crouzeix-Raviart element, but that can not implies an explicit bounds for the finite element error. The explicit a priori error estimates obtained in this paper provide computable error bounds and can serve as a posteriori error estimates for finite element methods [1, 35]. Furthermore, our explicit error estimates for the mixed and nonconforming elements are consistent with the maximal angle condition as the conforming linear Lagrange triangular element [5, 22].

The rest of the paper is organized as follows. In section 2, we introduce the set-up and approximation of the model problem along with some notations and preliminary results for subsequent use. Section 3 presents the an explicit priori error estimate for the lowest order Raviart-Thomas finite element. Similar estimates are extended to the nonconforming Crouzeix-Raviart element in section 4. Some numerical experiments are carried out in section 5. Finally, some comments and extensions of the results are given in section 6.

2. An Explicit Bound of the Inf-Sup Constant

In this section, after recalling the model formulation and some notation, we give a sharp Friedrichs' type inequality, based on which we obtain an explicit bound of the inf-sup constant.

Throughout this paper, we denote with small letters the scalar functions, with small bold fonts the vectorial ones. We will adopt the standard conventions for Sobolev norms and semi-norms of a function v defined on an open set G: