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A NEW APPROACH TO RECOVERY OF DISCONTINUOUS GALERKIN*

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Abstract

A new recovery operator $P : Q_n^{disc}(\mathcal{T}) \to Q_{n+1}^{disc}(\mathcal{M})$ for discontinuous Galerkin is derived. It is based on the idea of projecting a discontinuous, piecewise polynomial solution on a given mesh $\mathcal{T}$ into a higher order polynomial space on a macro mesh $\mathcal{M}$. In order to do so, we define local degrees of freedom using polynomial moments and provide global degrees of freedom on the macro mesh. We prove consistency with respect to the local L_2-projection, stability results in several norms and optimal anisotropic error estimates. As an example, we apply this new recovery technique to a stabilized solution of a singularly perturbed convection-diffusion problem using bilinear elements.

Mathematics subject classification: 65N12, 65N15, 65N30.
Key words: Discontinuous Galerkin, Postprocessing, Recovery.

1. Introduction

The importance of developing superconvergence recovery techniques for finite element approximations is two folded: firstly, the objective is to improve the approximation accuracy of low order finite elements on coarse meshes, which will significantly reduce the computational costs to achieve a certain accuracy. Secondly, the recovered solution values can be used in computation of a posteriori error estimators, which are essential for estimating the accuracy of finite element approximations and for guiding the mesh refinement in adaptive methods.

The main objective in this paper is the improvement of solution accuracy by using supercloseness results and an appropriate recovery technique (postprocessing). This type of superconvergence by recovery is well-known and has been extensively studied in the literature for different classes of problems, see, e.g., [5, 6, 9, 16]. The application of this technique to stabilized finite element discretization for solving singularly perturbed problems can be found in [11, 13, 14]. It has been shown that (in the two-dimensional case) the vertex-edge-cell interpolant, studied in [1], is superclose to the streamline-diffusion finite element solution on a Shishkin mesh. A recovery operator which is consistent with this special interpolant, allows to prove a superconvergence result for the postprocessed SDFEM solution.

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An alternative stabilization method for singularly perturbed problems is the discontinuous Galerkin method for which a supercloseness result with respect to the discontinuous, local L_2-projection onto piecewise bilinear functions has been established in [12]. The discussed method therein is the so called NIPG, see also [3, 10].

Our recovery techniques applies to the more general case of the local L_2-projection onto the space of discontinuous, piecewise polynomials of arbitrary degree $n \in \mathbb{N}$ in each variable and in any space dimension. Therefore it can be applied to a more general class of discontinuous Galerkin methods. We choose for application the NIPG because here a supercloseness result is known. For convection-diffusion equations in 1d several supercloseness results using numerical traces and possible postprocessing methods are known, see [4, 15].

We also recommend the reader to the recent reference [18] on recovery techniques in finite elements with special emphasis on Zienkiewicz-Zhu's patch recovery and polynomial preserving recovery.

The outline of this article is as follows. We start in Section 2 with the $1d$-recovery operator. In Section 3 we construct the $2d$-recovery operator and prove stability and anisotropic error estimates. Finally, in Section 4 we connect our results to recently published results [12] in the case of bilinears on a Shishkin mesh for a singularly perturbed partial differential equation.

Notation: For a function $u : \mathcal{T} \to \mathbb{R}$ which belongs piecewise in L_2 we define the broken L_2-norm by

$$\|u\|_{0,\mathcal{T}} = \left(\sum_{K \in \mathcal{T}} \|u\|_{0,K}^2 \right)^{1/2}.$$

2. Basics in $1d$

We start the definition of the recovery operator in one space-dimension. In order to simplify the notation we will work on reference elements. Thus, let $I_L := [-1, 0]$ and $I_R := [0, 1]$ be the reference intervals.

Our operator will be a projection onto a higher order polynomial space on macro meshes. Let $I_M := I_L \cup I_R$ denote the reference macro element to a given macro element consisting of two intervals. The reference mesh consists of the two subintervals of I_M and is denoted by $\mathcal{T} := \{I_L, I_R\}$.

We start the definition of the projector by defining local degrees of freedom on this mesh. Let

$$R_i(v) := \int_0^1 \eta_i(t) v(t)\,\mathrm{d}t \quad \text{and} \quad L_i(v) := \int_{-1}^0 \eta_i(t+1) v(t)\,\mathrm{d}t, \quad \forall i = 0, \ldots, n \tag{2.1}$$

with $\{\eta_i\}_{i=0}^n$ denoting the Legendre polynomial basis of $\mathcal{P}_n(I_R)$, the space of polynomials of degree at most n. Due to the L_2-orthogonality of these polynomials, the sets $\{R_i\}_{i=0}^m$ and $\{L_i\}_{i=0}^m$ with $0 \le m \le n$ are $\mathcal{P}_m(I_R)$- resp. $\mathcal{P}_m(I_L)$-unisolvent, i.e. an element $v \in \mathcal{P}_m(I)$ is uniquely defined for given values $\{N_i^1 v\}_{i=0}^m$. Then, there is a local basis $\{\psi_i^1\}_{i=0}^n$ of $\mathcal{P}_n(I_R)$ with

$$R_i(\psi_j^1) = \delta_{ij},\; i, j = 0, \ldots, n \tag{2.2}$$

where δ_{ij} is the Kronecker delta. Clearly our local basis functions are scaled Legendre polynomials with $\deg \psi_i^1 = i$, $i = 0, \ldots, n$, and the interpolation operator defined by

$$\pi v \in P_n^{disc}(\mathcal{T}) : R_i(\pi v) = R_i(v), \quad L_i(\pi v) = L_i(v), \quad i = 0, \ldots, n \tag{2.3}$$