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ON THE OPTIMIZATION OF EXTRAPOLATION METHODS FOR SINGULAR LINEAR SYSTEMS*

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Abstract

We discuss semiconvergence of the extrapolated iterative methods for solving singular linear systems. We obtain the upper bounds and the optimum convergence factor of the extrapolation method as well as its associated optimum extrapolation parameter. Numerical examples are given to illustrate the theoretical results.

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Key words: Singular systems, Extrapolated iterative method, Semiconvergence, Optimum value.

1. Introduction

Consider a system of linear equations

$$Ax = b, \tag{1.1}$$

where $A \in \mathcal{C}^{n\times n}$ is singular, b, $x \in \mathcal{C}^n$ with b known and x unknown. We assume that the linear system (1.1) is solvable, i.e., it has at least one solution. In order to solve the linear system (1.1) with iterative methods, the coefficient matrix A is split into

$$A = M - N, \tag{1.2}$$

where M is nonsingular. Then a linear stationary iterative method for solving (1.1) can be described as follows.

$$x^{k+1} = Tx^k + M^{-1}b, \quad k = 0, 1, 2, \cdots, \tag{1.3}$$

where $T = M^{-1}N$ is the iteration matrix.

The iterative method (1.3) is called semiconvergent if for every x^0 the sequence defined by (1.3) converges to a solution of (1.1). It is well known that the iterative method (1.3) is semiconvergent if and only if the pseudo-spectral radius

$$\vartheta(T) \equiv \max\{|\mu|, \ \ \mu \in \sigma(T)\backslash\{1\}\}$$

is less than 1 and the elementary divisors associated with $\mu = 1 \in \sigma(T)$ are linear, i.e.,

$$\text{index}(I - T) = 1,$$

where $\sigma(T)$ denotes the spectrum of T and index(B) denotes the index of the matrix B, i.e., the smallest nonnegative integer k such that $\text{rank}(B^{k+1}) = \text{rank}(B^k)$ (rank(B) means the rank of

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B). In this case, the splitting (1.2) is also called semiconvergent and T is called a semiconvergent matrix. The associated convergence factor of T and the iterative method (1.3) is $\vartheta(T)$.

The semiconvergence of splitting (1.2) has been investigated by many papers (cf. [4, 11] and the references therein).

Moreover, some new results have been obtained by using matrix splittings and iterative methods to solve the linear complementarity problem (cf. [1, 2, 3, 17]).

For $\omega \in \mathcal{C}$ the extrapolation method of (1.3) can be defined by

$$x^{k+1} = T_\omega x^k + \omega M^{-1} b, \quad k = 0, 1, 2, \cdots, \tag{1.4}$$

where

$$T_\omega = (1 - \omega)I + \omega T$$

is the iteration matrix and ω is called the extrapolation parameter (cf. [8]). Clearly, if $\omega = 0$ then $T_0 = I$, which leads to a trivial case. Thus, we assume that $\omega \neq 0$.

Now, we assume that

$$A = D - Q, \tag{1.5}$$

where $D = diag(a_{11}, \cdots, a_{nn})$ is nonsingular. Associated with the splitting (1.5), the Jacobi iteration matrix J can be expressed as

$$J = D^{-1}Q.$$

The extrapolated Jacobi method is also called JOR method (cf. [16]) with the iteration matrix J_ω, namely,

$$J_\omega = (1 - \omega)I + \omega J.$$

The method (1.4) is consistent with (1.1) and is used to accelerate the convergence of the method (1.3). The extrapolation method for solving the singular systems has been discussed in many papers (cf. [7, 10, 12]).

Now, an interesting, important and also complicated problem is the determination of the optimum value ω_{opt} for ω, which minimizes $\vartheta(T_\omega)$. This problem has been discussed extensively by some researchers. It was treated by the geometrical method in [7].

In this paper, the determination of the sharp analytical upper bounds for $\min_\omega \vartheta(T_\omega)$ is achieved by an algebraic approach, which generalize the results in [15] to the singular case. On the other hand, these bounds are obtained for the good analytical values for the extrapolation parameter which coincide with the optimum ones under some additional conditions. In the theory presented no knowledge of the eigenvalues of T is required. Finally, some applications and numerical examples are given which support the theory developed. The paper is organized as follows. After establishing the bounds for $\min_\omega \vartheta(T_\omega)$ in Section 2, we extend the extrapolation theorem given in [6, 15] to the singular system and improve the corresponding results in [10, 12]. In Section 3, an application and the numerical results are given to illustrate the results presented in Sections 2.

2. Determination of Upper Bounds and Optimum Values

Lemma 2.1. ([10]) *For the singular linear system* (1.1) *the following results hold:*