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A POSTERIORI ESTIMATOR OF NONCONFORMING FINITE ELEMENT METHOD FOR FOURTH ORDER ELLIPTIC PERTURBATION PROBLEMS*

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Abstract

In this paper, we consider the nonconforming finite element approximations of fourth order elliptic perturbation problems in two dimensions. We present an *a posteriori* error estimator under certain conditions, and give an h-version adaptive algorithm based on the error estimation. The local behavior of the estimator is analyzed as well. This estimator works for several nonconforming methods, such as the modified Morley method and the modified Zienkiewicz method, and under some assumptions, it is an optimal one. Numerical examples are reported, with a linear stationary Cahn-Hilliard-type equation as a model problem.

Mathematics subject classification: 65N30.
Key words: Fourth order elliptic perturbation problems, Nonconforming finite element method, A posteriori error estimator, Adaptive algorithm, Local behavior.

1. Introduction

The parabolic perturbation problems, such as the Cahn-Hilliard-type equations, are frequently encountered in applications, see, e.g., [12, 14, 20]. Their stationary formations, namely the corresponding elliptic perturbation problems, are important for both theoretical analysis and computation. The numerical solution to such problems has been an interesting and practical topic in computational mathematics. Various finite element methods, both standard and nonstandard, have been developed for this problem, and their convergences were proven; see, e.g., [16, 17, 25, 30].

The adaptive finite element methods, in particular the h-version methods, are very useful for efficient numerical solutions. As to these methods, the key features are *a posteriori* error estimation and the strategy of mesh refinement. The *a posteriori* error estimation can be treated as an indicator of the distribution of the error on certain mesh. According to the *a posteriori* error estimation, the numerical solution can be carried out in the local, parallel or adaptive ways, see, e.g., [32]. In all these methods, an *a posteriori* error estimator is utilized as an indicator of the quality of the mesh.

It is pointed by Bank [9] that the notion of using *a posteriori* error estimates to measure and control the error in practical finite element calculations was first suggested by Babuska and Rheinboldt [5]. The approach in [5] provided the earliest general way for *a posteriori* error estimation with firm theoretical foundations. So far, *a posteriori* error estimation for conforming finite element methods, especially on second order problems, has been the subject of extensive investigation, see, e.g., [2, 5, 6, 9–12, 23, 33], the reviews [3, 8, 19, 21, 22] and the

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monographs [4, 7, 27]. However, the treatment of nonconforming methods has been subjected to sporadic attention. Dari et al. [18] considered the error as a combination of a conforming part and a nonconforming part, where the nonconforming part is estimated via the difference between the nonconforming solution and its smooth approximation. The idea has been carried out on the second order problems, with the help of the orthogonal decomposition (*Helmholtz* decomposition) of L^2. According to this, many ways were put forward to extend the method for conforming methods to nonconforming ones; see, e.g., [1] and the references therein. Castensen et al. [15] followed the idea and developed a technique to present a framework of *a posteriori* estimation for a class of nonconforming methods on parallelogram meshes. The framework has been shown to be effective for problems of second order. Even though the *a posteriori* estimation for fourth order problems can date back to [27], however, partially because that few nonconforming finite element spaces contain a subspace consisting of C^1 continuous functions, there are few works dealing with the nonconforming methods directly. A general framework of *a posteriori* error estimation is presented in [26] for the nonconforming methods, and it is shown that the methodology of decomposing the errors can be used for problems with arbitrarily high order.

The error estimators obtained in such ways give upper bounds of the global error, and can be computed in *a posteriori* way. However, in local sense, they may provide upper bounds of error as well as mesh indicators. Xu and Zhou [32] showed a local upper bound for *conforming* methods applied to second order problems. Wang and Zhang [31] proved that a local *a posteriori* error estimator can be a local upper bound of the error up to higher order terms for the *nonconforming* finite element methods to two dimensional biharmonic equations.

In this paper, we study the *a posteriori* error estimation for nonconforming finite element methods for the elliptic perturbation problems. A two dimensional linear stationary Cahn-Hilliard-type equation is used as a model problem. The rest of the paper is organized as follows. In Section 2, some preliminary materials are provided. In Section 3, global *a posteriori* error estimator is obtained for general nonconforming finite element discretization methods on shape-regular grids for the model problem. The deduction uses the same idea of the framework [26]. The efficiency of the estimator is devised and analyzed in Section 4. Based upon certain convergence assumptions, the estimator is optimal in the sense that the *a posteriori* error estimator has the same convergence order as that of the *a priori* error estimator. An h-version adaptive method is discussed and some numerical experiments are reported in Section 5. In the final section, further discussions are presented and the local behavior of the estimator is analyzed.

2. Preliminaries

In this section, we describe the model problem and the corresponding nonconforming finite element methods.

Let Ω be a bounded domain in R^2, with the boundary $\partial\Omega$ and ν the unit outer normal vector to $\partial\Omega$. For nonnegative integer s, we shall use the standard notation $H^s(\Omega)$ for Sobolev space, $\|\cdot\|_{s,\Omega}$ the associated norm and $|\cdot|_{s,\Omega}$ the associated seminorm. We shall omit s and not distinguish the norm and the seminorm when $s=0$. In addition we denote

$$H_0^1(\Omega)=\{v\in H^1(\Omega): v|_{\partial\Omega}=0\},\quad H_0^2(\Omega)=\left\{v\in H^2(\Omega)\cap H_0^1(\Omega): \frac{\partial v}{\partial\nu}\big|_{\partial\Omega}=0\right\}.$$