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A METHOD FOR SOLVING THE INVERSE SCATTERING PROBLEM FOR SHAPE AND IMPEDANCE*

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Abstract

The inverse problem considered in this paper is to determine the shape and the impedance of an obstacle from a knowledge of the time-harmonic incident field and the phase and amplitude of the far field pattern of the scattered wave in two-dimension. Single-layer potential is used to approach the scattered waves. An approximation method is presented and the convergence of the proposed method is established. Numerical examples are given to show that this method is both accurate and easy to use.

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Key words: Impedance boundary condition, Helmholtz equation, Inverse scattering, Convergence.

1. Introduction

The inverse scattering problem for time-harmonic acoustic waves in two-dimension has been considered for various boundary conditions in a series of papers [1–7]. Among these problems, we are interested in numerical methods for determining the shape and the impedance of an obstacle from the knowledge of the incident field and the scattered field of the far field pattern.

Let D be a bounded, connected domain in the plane with boundary $\partial D \in C^2$ and let the incident field u^i be given by $u^i(x) = \exp[\mathrm{i}kx \cdot d]$ where $k > 0$ is the wave number and d is a fixed unit vector. If we denote the scattered field by u^s and define the total field u by $u = u^i + u^s$, then the direct scattering problem is to find a solution $u \in C^2(\mathbb{R}^2 \backslash \bar{D}) \cap C(\mathbb{R}^2 \backslash D)$ of the Helmholtz equation

$$\Delta_2 u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{D}, \tag{1.1}$$

which satisfies the boundary condition

$$\frac{\partial u}{\partial \nu} + \mathrm{i}k\lambda(x)u = 0 \quad \text{on } \partial D; \tag{1.2}$$

and u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r} \left\{ \frac{\partial u^s}{\partial r} - \mathrm{i}ku^s \right\} = 0, \quad r = |x|, \tag{1.3}$$

uniformly in all directions $x/|x|$.

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Under the above conditions, it is easily shown [8] that u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}}\{u_\infty(\hat{x}) + \mathcal{O}(|x|^{-1})\}, \quad |x| \to \infty, \tag{1.4}$$

where u_∞ is known as the far field pattern of the scattered wave u^s. From Green's formula and the asymptotic behavior of the Hankel function $H_0^{(1)}$, we can easily show [8] that

$$u_\infty(\hat{x}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} \left\{ u^s(y)\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial u^s}{\partial \nu}(y)e^{-ik\hat{x}\cdot y} \right\} ds(y), \tag{1.5}$$

for $\hat{x} = x/|x|$.

For the problem (1.1)-(1.3), there exists the following theorem.

Theorem 1.1. ([9]) *The exterior impedance boundary-value problem has at most one solution provided $Im(\lambda) \geq 0$ on ∂D. The solution u^s in $\mathbb{R}^2\backslash D$ and each differentiation of u^s in $\mathbb{R}^2\backslash \bar{D}$ depend continuously on the boundary data.*

Let $\Gamma \in C^2$ be a closed curve contained in D and assume that k^2 is not a Dirichlet eigenvalue of Laplacian in Γ. Let the single-layer potential

$$v(x) = \int_\Gamma \varphi(y)\Phi(x,y)ds(y), \quad \varphi \in L^2(\Gamma) \tag{1.6}$$

approach the scattered field u^s, where $\Phi(x,y) = \frac{i}{4}H_0^{(1)}(k\,|x-y|)$ denotes the fundamental solution to the Helmholtz equation in two-dimension. From the asymptotic for $u(x)$:

$$u(x) = \frac{e^{ik|x|}}{\sqrt{|x|}}\{u_\infty(\hat{x}) + \mathcal{O}(|x|^{-1})\}, \quad |x| \to \infty,$$

uniformly in all directions $\hat{x} = x/\,|x|$, and the asymptotic for the Hankel function:

$$H_0^{(1)}(r) = \sqrt{\frac{1}{\pi r}}e^{i(r-\pi/4)}\Big(1 + o(\frac{1}{r})\Big),$$

we see that the far-field pattern of the potential (1.6) is given by

$$u_\infty(\hat{x}) = \frac{e^{-i\pi/4}}{\sqrt{8\pi k}} \int_\Gamma ie^{-ik\hat{x}\cdot y}\varphi(y)ds(y). \tag{1.7}$$

To solve inverse obstacle scattering problems, we consider a numerical method to solve the inverse scattering problem for shape and impedance. Hence, for the given far-field pattern, we solve the integral equation

$$(F\varphi)(\hat{x}) = u_\infty(\hat{x}), \tag{1.8}$$

where $F : L^2(\Gamma) \to L^2(\Omega)$ is defined by

$$(F\varphi)(\hat{x}) := \frac{e^{-i\pi/4}}{\sqrt{8\pi k}} \int_\Gamma ie^{-ik\hat{x}\cdot y}\varphi(y)ds(y), \quad \hat{x} \in \Omega. \tag{1.9}$$

Then we need to find the boundary Γ as the location where the boundary condition (1.2) is satisfied in a least square sense.

In comparison with [10], our reconstructions do not require the solution of the function u and its normal derivative $\partial u/\partial \nu$ at each iteration step. We only require the nonzero initials of φ, ρ, λ. Furthermore, this method does not contain the hyper-singular operator, which makes the computation easy.