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A MIXED FINITE ELEMENT METHOD ON A STAGGERED MESH FOR NAVIER-STOKES EQUATIONS*

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Abstract

In this paper, we introduce a mixed finite element method on a staggered mesh for the numerical solution of the steady state Navier-Stokes equations in which the two components of the velocity and the pressure are defined on three different meshes. This method is a conforming quadrilateral $Q_1 \times Q_1 - P_0$ element approximation for the Navier-Stokes equations. First-order error estimates are obtained for both the velocity and the pressure. Numerical examples are presented to illustrate the effectiveness of the proposed method.

Mathematics subject classification: 35Q30, 74G15, 74S05.
Key words: Mixed finite element method, Staggered mesh, Navier-Stokes equations, Error estimate.

1. Introduction

It is well known that the simplest conforming low-order elements like the $P_1 - P_0$ (linear velocity vector, constant pressure) triangular element and $Q_1 - P_0$ (bilinear velocity vector, constant pressure) quadrilateral element are not stable when applied to the Navier-Stokes (NS) equations [6]. Therefore, some special treatments are needed in order to keep the schemes stable. During the last two decades, there has been a rapid development in practical stabilization technique for the $P_1 - P_0$ element and the $Q_1 - P_0$ element for solving the NS equations [1, 7, 8, 9, 11]. In [3], an economical finite element scheme is proposed to construct three finite-dimensional subspaces for the two velocity components and the pressure. In [2], a mixed finite element scheme for the Stokes equations is investigated. In this paper, we extend the idea in [3] to construct a mixed finite element scheme for the NS equations, which is more efficient than the scheme given in [3] as the degree of freedom is reduced. The optimal error estimate of this scheme is obtained.

The outline of the paper is as follows. In the next section, we give a formulation of the mixed finite element method for the Navier-Stokes equations. In Section 3, the error estimates will be provided. In Section 4, two numerical examples will be considered. Finally, we end the paper with a short concluding section.

2. A Mixed Finite Element Formulation for the NS Equations

We consider the following boundary value problem of the Navier-Stokes equations:

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \text{div } \mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

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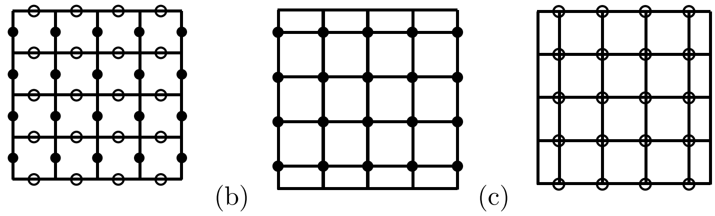

(a) (b) (c)

Fig. 2.1. Quadrangulations: (a) $\mathcal{J}_h$, (b) $\mathcal{J}_h^1$, (c) $\mathcal{J}_h^2$.

where $\Omega \subset \mathrm{R}^2$ is a rectangular domain, ν is the viscosity, $\mathbf{u} = (u_1, u_2)^T$ represents the velocity vector, p is the pressure, and $\mathbf{f} = (f_1, f_2)^T$ is the given body force. Let $H^n(\Omega)$ and $H_0^1(\Omega)$ denote the standard Sobolev spaces with the norm $\|\cdot\|_{n,\Omega}$ and $\|\cdot\|_{1,\Omega}$ respectively. Furthermore, let

$$\mathbf{V} \equiv H_0^1(\Omega) \times H_0^1(\Omega), \quad M \equiv \left\{ q : q \in L^2(\Omega) \text{ and } \int_\Omega q dx = 0 \right\}.$$

Then the boundary value problem (2.1) is reduced to the following equivalent variational problem [3]:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{V} \text{ and } p \in M, \text{ such that} \\ a(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}; \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) = 0 & \forall q \in M, \end{cases} \tag{2.2}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \nu \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} dx,$$

$$a_1(\mathbf{w}; \mathbf{u}, \mathbf{v}) = \frac{1}{2} \sum_{i,j=1}^{2} \int_\Omega w_j \Big(\frac{\partial u_i}{\partial x_j} v_i - \frac{\partial v_i}{\partial x_j} u_i \Big) dx,$$

$$b(\mathbf{v}, q) = -\int_\Omega q \text{div } \mathbf{v} dx, \quad (\mathbf{f}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx.$$

For simplicity we assume that the domain Ω is a unit square, but the finite element method discussed below can be easily generalized to include the case that the domain Ω is rectangular. Let N be a given integer and $h = 1/N$. We shall construct the finite-dimensional subspaces of $\mathbf{V}$ and M by introducing three different quadrangulations $\mathcal{J}_h$, $\mathcal{J}_h^1$, $\mathcal{J}_h^2$ of Ω. First we divide Ω into equal squares

$$T_{i,j} = \Big\{ (x_1, x_2) : (x_1)_{i-1} \le x_1 \le (x_1)_i,\ (x_2)_{j-1} \le x_2 \le (x_2)_j \Big\}, \quad i, j = 1, \cdots, N,$$

where $(x_1)_i = ih$ and $(x_2)_j = jh$. The corresponding quadrangulation is denoted by $\mathcal{J}_h$. Then for all $T_{i,j} \in \mathcal{J}_h$ we connect all the midpoints of the vertical sides of $T_{i,j}$ by straight line segments if the midpoints have a distance h, and extend the resulting mesh to the boundary Γ. Then Ω is divided into squares and rectangles, and the corresponding quadrangulation is denoted by $\mathcal{J}_h^1$. Similarly, for all $T_{i,j} \in \mathcal{J}_h$ we connect all the midpoints of the horizontal sides of $T_{i,j}$ by straight line segments if the midpoints have a distance h, and extend the resulting mesh to the boundary Γ. Then we obtained the third quadrangulation of Ω, which is denoted by $\mathcal{J}_h^2$ (see Fig. 2.1).