Journal of Computational Mathematics, Vol.25, No.1, 2007, 2–12.

A BOUNDARY INTEGRAL METHOD FOR COMPUTING ELASTIC MOMENT TENSORS FOR ELLIPSES AND ELLIPSOIDS *1)

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Abstract

The concept of elastic moment tensor occurs in several interesting contexts, in particular in imaging small elastic inclusions and in asymptotic models of dilute elastic composites. In this paper, we compute the elastic moment tensors for ellipses and ellipsoids by using a systematic method based on layer potentials. Our computations reveal an underlying elegant relation between the elastic moment tensors and the single layer potential.

Mathematics subject classification: 45A05, 74B05.
Key words: Elastic moment tensor, Explicit formulae, Integral equations.

1. Introduction

Let B be a bounded Lipschitz domain in $\mathbb{R}^d, d = 2, 3$. Assume that the Lamé parameters of B are given by $(\widetilde{\lambda}, \widetilde{\mu})$, while those of the background $\mathbb{R}^d \setminus \overline{B}$ are given by (λ, μ). Attached to the inclusion B is a 4-tensor m_{pq}^{ij}, $i, j, p, q = 1, \ldots, d$, called the elastic moment tensor (EMT), or the elastic polarization tensor. The notion of EMT can be most simply described in the following manner. Denote by $\mathcal{L}_{\lambda,\mu}$ the Lamé operator associated with the parameters λ and μ and consider $\mathbf{H}$ to be a vector-valued function satisfying $\mathcal{L}_{\lambda,\mu}\mathbf{H} = 0$ in $\mathbb{R}^d$, $d = 2, 3$. If the field $\mathbf{H}$ is perturbed due to the presence of an elastic inclusion B with the Lamé parameters $(\widetilde{\lambda}, \widetilde{\mu})$ then the ith component of the perturbation is given by

$$\sum_{j=1}^{d} \sum_{p,q=1}^{d} \partial_p H_j(0) \partial_q \mathbf{\Gamma}(x) m_{pq}^{ij} + O(|x|^{1-d}) \quad \text{as } |x| \to \infty, \tag{1.1}$$

where H_j is the jth component of $\mathbf{H}$ and $\mathbf{\Gamma}$ is the fundamental solution to the Lamé equation with the Lamé parameters λ, μ. See [1] for a rigorous derivation of formula (1.1) which shows that through the elastic moment tensor, $M = (m_{pq}^{ij})$, we have a complete information about the leading-order term in the far-field expansion of $\mathbf{H}$. See also [6] for a representation of the perturbation by Elsheby's tensor. It is worth mentioning that the use of the EMT leads to stable and accurate algorithms for the numerical computations of the displacement field in the presence of small elastic inclusions. It is known that small size features cause difficulties in the numerical solution of the problem by the finite element or finite difference methods. This is because such features require refined meshes in their neighborhoods, with their attendant problems.

The notion of EMT also occurs naturally in several other physical contexts, in particular in asymptotic expansions of perturbations of the elastic energy [10, 11] and in models of the

* Received January 4, 2006; accepted February 16, 2006.
1) Partly supported by Korea Science and Engineering Foundation grant R02-2003-000-10012-0.

effective properties of dilute elastic composites [7, 8, 12, 13]. Recently, the EMT has been used in the inverse problem of reconstructing diametrically small elastic inclusions from boundary measurements [1, 2, 4, 8]. It turns out that we can determine the EMT of the inclusion via boundary measurements. Since EMT carries fairly good information about the size of the inclusion, the volume of the elastic inclusion can be estimated by means of the EMT.

The purpose of this paper is to present a quite general and elegant method for computing the EMT based on layer potential techniques. In particular, we apply this method to provide explicit formulae for the EMTs associated with ellipses and ellipsoids. The method reveals an interesting relation between the single layer potentials and the EMT. The EMT for ellipses has been computed in [1] using a complex analysis representation of the solutions to the two-dimensional Lamé system [14]. The method of this paper is completely different from that in [1]. Moreover, it enables us to find explicit formula for the EMT for ellipsoids. It should be mentioned that a quite similar method has been used to compute the polarization tensor associated to the conductivity problem for ellipses and ellipsoids [9].

This paper is organized as follows. In Section 2 we review the definition of the EMT in terms of layer potentials and present a general scheme to compute it. Sections 3 and 4 are devoted to the derivation of EMTs for ellipses and ellipsoids.

2. Single Layer Potential and Elastic Moment Tensor

Let $\mathbf{\Gamma} = (\Gamma_{ij})_{i,j=1}^d$ be a fundamental solution to the Lamé system, namely,

$$\Gamma_{ij}(x) := \begin{cases} -\dfrac{A}{4\pi}\dfrac{\delta_{ij}}{|x|} - \dfrac{B}{4\pi}\dfrac{x_i x_j}{|x|^3} & \text{if } d=3, \\ \dfrac{A}{2\pi}\delta_{ij}\ln|x| - \dfrac{B}{2\pi}\dfrac{x_i x_j}{|x|^2} & \text{if } d=2, \end{cases} \qquad x \neq 0,$$

where

$$A = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{2\mu+\lambda}\right) \quad \text{and} \quad B = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{2\mu+\lambda}\right).$$

Let the constants (λ, μ) denote the background Lamé coefficients, that are the elastic parameters in the absence of any inclusions. Let B be a bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$. Assume that B has the pair of Lamé constants $(\widetilde{\lambda}, \widetilde{\mu})$ which is different from that of the background elastic body, (λ, μ). It is always assumed that

$$\mu > 0, \quad d\lambda + 2\mu > 0, \quad \widetilde{\mu} > 0 \quad \text{and} \quad d\widetilde{\lambda} + 2\widetilde{\mu} > 0.$$

We also assume that

$$(\lambda - \widetilde{\lambda})(\mu - \widetilde{\mu}) \geq 0, \quad ((\lambda - \widetilde{\lambda})^2 + (\mu - \widetilde{\mu})^2 \neq 0).$$

The single layer potential of the density function φ on B associated with the Lamé parameters (λ, μ) is defined by

$$\mathcal{S}_B\varphi(x) := \int_{\partial D} \mathbf{\Gamma}(x-y)\varphi(y)\, d\sigma(y), \quad x \in \mathbb{R}^d.$$

Analogously, we denote by $\widetilde{\mathcal{S}}_B$ the single layer potential on ∂B corresponding to the Lamé constants $(\widetilde{\lambda}, \widetilde{\mu})$.

The following jump relation is well-known:

$$\frac{\partial(\mathcal{S}_B\varphi)}{\partial\nu}\Big|_+ - \frac{\partial(\mathcal{S}_B\varphi)}{\partial\nu}\Big|_- = \varphi \quad \text{a.e. on } \partial B,$$

where $\partial\mathbf{u}/\partial\nu$ denotes the conormal derivative, *i.e.*,

$$\frac{\partial\mathbf{u}}{\partial\nu} := \lambda(\nabla\cdot\mathbf{u})N + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)N \quad \text{on } \partial B.$$

Here N is the outward unit normal to ∂B and the superscript T denotes the transpose of a matrix.