Journal of Computational Mathematics, Vol.25, No.5, 2007, 553–560.

STRUCTURES OF CIRCULANT INVERSE M-MATRICES *

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Abstract

In this paper, we present a useful result on the structures of circulant inverse M-matrices. It is shown that if the $n\times n$ nonnegative circulant matrix $A = Circ[c_0, c_1, \cdots, c_{n-1}]$ is not a positive matrix and not equal to $c_0 I$, then A is an inverse M-matrix if and only if there exists a positive integer k, which is a proper factor of n, such that $c_{jk} > 0$ for $j = 0, 1, \cdots, [\frac{n-k}{k}]$, the other c_i are zero and $Circ[c_0, c_k, \cdots, c_{n-k}]$ is an inverse M-matrix. The result is then extended to the so-called generalized circulant inverse M-matrices.

Mathematics subject classification: 15A48, 15A29, 15A57.
Key words: Nonnegative matrices, Circulant matrix, Inverse M-matrices.

1. Introduction

A real matrix A is called positive (nonnegative), denoted by $A > 0$ ($A \geq 0$), if every entry $a_{i,j}$ is positive (nonnegative). A real matrix is called a Z-matrix if all its off-diagonal entries are nonpositive. A nonnegative square matrix is called an inverse M-matrix if it is invertible and its inverse is a Z-matrix.

A square matrix A is called reducible if there is a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are non-empty square matrices. A matrix is irreducible if it is not reducible.

The following lemmas, which will be used later, involve zero and nonzero pattern or structures of inverse M-matrices.

Lemma 1.1. (Corollary 2.2 in [10]) *If A is an irreducible inverse M-matrix, then A is positive.*

Lemma 1.2. ([6]) *Suppose that A is an inverse M-matrix, let k be a positive integer. Then the (i,j) entry of A^k is zero if and only if the (i,j) entry of A is zero.*

Lemma 1.3. *Let A be a partitioned inverse M-matrix:*

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,r} \\ A_{2,1} & A_{2,2} & \dots & A_{2,r} \\ \dots & \dots & \dots & \dots \\ A_{r,1} & A_{r,2} & \dots & A_{r,r} \end{bmatrix}.$$

* Received December 15, 2005; final revised September 1, 2006; accepted October 1, 2006.

Assume that $A_{i,i}$ $(i = 1, 2, \ldots, r)$ *are positive square matrices. Then* $A_{i,j}$ *also is positive if* $A_{i,j} \neq 0$ *when* $i \neq j$*.*

Proof. Let A^k have the same partition as A and denote the (i, j) block of A^k by $A^{(k)}_{i,j}$. If $A_{i,j} \neq 0$ for some $i \neq j$, then

$$A^{(2)}_{i,j} = \sum_{l=1}^{r} A_{i,l}A_{l,j} \geq A_{i,i}A_{i,j} + A_{i,j}A_{j,j}.$$

Since $A_{i,i}$, $A_{j,j}$ are positive and $A_{i,j}$ is nonnegative, we know from the inequality that $A^{(2)}_{i,j}$ has at least one positive row and one positive column. Thus

$$A^{(3)}_{i,j} = \sum_{l=1}^{r} A^{(2)}_{i,l}A_{l,j} \geq A^{(2)}_{i,i}A_{i,j} + A^{(2)}_{i,j}A_{j,j}$$

must be positive. By Lemma 1.2, $A_{i,j}$ is positive.

A matrix C is called a circulant matrix if it is of the form:

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_2 & \cdots & c_{n-1} & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix} \qquad (1.1)$$

We will denote the circulant matrix C in (1.1) by $Circ[c_0, c_1, \cdots, c_{n-1}]$ for notational convenience.

Inverse M-matrices and circulant matrices are two classes of important matrices. Inverse M-matrices often occur in systems of linear or non-linear equations or eigenvalues problems in a wide variety of areas including finite difference methods for partial differential equations, input-output production and growth models in economics, iterative methods in numerical analysis, and Markov processes in probability and statistics. A number of properties of inverse M-matrices have been given in [1], [6]-[9]. Circulant matrices are often used as preconditioner for Toeplitz linear systems since they can be easily inverted and super-fast computed [2, 3].

In this paper, we present an interesting result on the structures of circulant inverse M-matrices. We show that a nonnegative but not positive circulant matrix $Circ[c_0, c_1, \cdots, c_{n-1}](\neq c_0 I)$ is an inverse M-matrix if and only if there exists a positive integer k, which is a proper factor of n, such that $c_{jk} > 0$ for $j = 0, 1, \ldots, [\frac{n-k}{k}]$, the other c_i $(i.e., i \neq jk)$ are zero and $Circ[c_0, c_k, \cdots, c_{n-k}]$ is an inverse M-matrix. The result is then extended to so-called generalized circulant inverse M-matrices.

In the next section, we review some definitions and basic properties of digraphs and introduce a new digraph we will use in this paper. Section 3 presents our main result. The result then is extended to so-called generalized circulant matrices in the last section.

2. Preliminaries

Let $< n >= \{1, 2, \ldots, n\}$. The digraph $G = (N, E)$ consists of the vertex set N, conveniently labeled from 1 to n, and the set of directed edges (arcs) $E = \{(i, j)|i, j \in N\}$. A path in a