NATURAL SUPERCONVERGENT POINTS
OF EQUILATERAL TRIANGULAR FINITE ELEMENTS
– A NUMERICAL EXAMPLE [*1)]

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Abstract

A numerical test case demonstrates that the Lobatto and the Gauss points are not natural superconvergent points of the cubic and the quartic finite elements under equilateral triangular mesh for the Poisson equation.

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1. Introduction

Natural superconvergent points are those points where the rate of convergence exceeds the best possible global rate without post-processing. Research in this area started in the early 70s, and may even trace back to the late 60s (see [5, 6, 7, 8, 9, 12, 14] and references therein). Consider the C^0 finite element approximation for second-order elliptic equations. It is well known that the Lobatto points and the Gauss points are superconvergent points for function values and derivatives, respectively. This result is valid for the one dimensional case, as well as for the tenser-product space in higher dimensional settings. As for triangular elements, the situation is much more complicated. Earlier researches focused only on lower-order elements, namely, the linear and the quadratic elements under strongly regular meshes.

In the mid-90s, two systematic methods to find superconvergent points were developed. One was the symmetry theory due to Schatz-Sloan-Wahlbin [11, 12]. By this theory, super-convergence occurs at mesh symmetry centers. One advantage of the symmetry theory is its generality, since it is applicable in all dimensions. Nevertheless, this theory does not say that there is no other superconvergence points, and therefore it is not conclusive. Almost at the same time, another approach was proposed by Babuška-Strouboulis *et al*[2, 3]. They established a theoretical framework which narrows the task of locating superconvergence points to finding intersections of some polynomial contours on a "master cell". The actual procedure was carried out by a computer algorithm without explicitly constructing those polynomials, and the computed superconvergent points were reported up to 10 digits. This approach is called the "computer-based proof". Using that computer algorithm, they predicted all derivative superconvergence points for the Poisson equation, the Laplace equation, and the linear elasticity equation under four different triangular mesh patterns for polynomial finite element spaces of degrees up to 7. The advantage of this approach is that it is conclusive.

Along the line of the computer-based proof, Zhang proposed an analytic approach which constructs explicitly the needed polynomials through an orthogonal decomposition [13]. Using

this approach, Lin-Zhang studied superconvergent points for triangular elements [10]. Their results verify that the computed data for triangular elements in [3] have 9 digits of accuracy except for one pair (with 8-7 accurate digits). In addition, they reported superconvergent points for function values, which are not discussed in the computer-based proof. The Lin-Zhang approach has the following advantages:

1. The superconvergent points are computed with high accuracy. With polynomials explicitly given, one can easily verify how accurate those points are.

2. It can be readily verified. With paper and pencil, an interested reader may check if those polynomials are indeed the periodic finite element solutions on master cells. As for the superconvergent points, when polynomials are explicitly given, a root finding can be done analytically for lower-order cases (which are the most interesting cases anyway).

3. It provides more insight. An orthogonal polynomial basis functions on triangular meshes is constructed in a systematic way, which reveals the structure of the periodic finite element solution.

4. It can be generalized to 3D cases, where superconvergence results are relatively scarce.

As a related work, Chen investigated superconvergent points in the triangular elements of the regular pattern using the orthogonal decomposition and element analysis technique [5]. As an example of his approach, superconvergent points in the cubic element are calculated for the Poisson equation as well as for the Laplace equation. These points are consistent with the symmetry theory and the computer-based proof.

According to the above review, investigation of natural superconvergent points for triangular elements seems to be over except for one special case, the equilateral triangulation. Yet, this is the most important case, since automatic mesh generators based on the Delaunay triangulation produce nearly equilateral triangles for most part of the mesh. There was an interesting result due to Blum-Lin-Rannacher in 1986 [4], which revealed that the convergent rate for linear element at element vertices is of $O(h^4)$ for equilateral mesh. This rate is two orders higher than that of the regular mesh pattern. Therefore, it is natural to ask if there are other special properties associated with equilateral triangular mesh for higher-order finite elements. Motivated with the fact that the Lobatto (Gauss) points in the quadratic element are supercovergent points of function (derivative) values along element edges (this is true for both regular and equilateral triangulation [1, 14]), one may wonder if it is also true for higher-order elements. The current paper provides a negative answer to this question. With a non-trivial numerical example, we demonstrate that neither the Lobatto nor the Gauss points are natural superconvergent points for the cubic and the quartic elements in the equilateral meshes. The convergence rate is the same for the regular and the equilateral triangulations with only one exception, the linear element.

2. A Numerical Example

Define an equilateral triangular domain Ω enclosed by the three straight lines $y = 0$, $y = \sqrt{3}x$, and $y = \sqrt{3}(1 - x)$. Consider the boundary value problem

$$-\Delta u = f \quad \text{in} \ \Omega, \qquad u = 0 \quad \text{on} \ \partial\Omega. \tag{2.1}$$

We choose f such that $u = y(y - \sqrt{3}x)(y + \sqrt{3}x - \sqrt{3})e^{x+y}$, and then solve (2.1) by the linear, the quadratic, the cubic, and the quartic finite elements under uniform triangulation of Ω by dividing each side of Ω to n sub-intervals. We set $h^{-1} = n = 8$ (see Figure 1) for the initial mesh, and use the regular refinement with bi-section strategy. Using this kind of meshes, we will solve (2.1) with MATLAB 6.5.1 and FEMLAB 2.3. Earlier versions of MATLAB may lead to different results as it uses Gaussian elimination to solve unrecognized sparse linear systems. MATLAB 6.5.1 has a new solver that supports more types of sparse linear systems including the systems arising in solving (2.1).

We calculate the error $e_h = u - u_h$ at the Lobatto points and $\partial_t e_h = \vec{t} \cdot \nabla(u - u_h)$ at the Gauss points. Here $\vec{t}$ is the tangential unit vector. Table 1 lists the Lobatto and the Gauss