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QUANTUM COMPLEXITY OF THE INTEGRATION PROBLEM FOR ANISOTROPIC CLASSES [*1)]

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Abstract

We obtain the optimal order of high-dimensional integration complexity in the quantum computation model in anisotropic Sobolev classes $W^{\mathbf{r}}_{\infty}([0,1]^d)$ and Hölder Nikolskii classes $H^{\mathbf{r}}_{\infty}([0,1]^d)$. It is proved that for these classes of functions there is a speed-up of quantum algorithms over deterministic classical algorithms due to factor n^{-1} and over randomized classical methods due to factor $n^{-1/2}$. Moreover, we give an estimation for optimal query complexity in the class $H^{\Lambda}_{\infty}(D)$ whose smoothness index is the boundary of some complete set in $\mathbb{Z}^d_+$.

Mathematics subject classification: 68Q01.
Key words: Quantum computation, Integration problem, Anisotropic classes, Complexity

1. Introduction

Quantum computers, whose basic operators are based on the theory of quantum mechanics, equip with the amazing computational speed which is much faster than that of classical computers. The questions arisen by the powerful conceptual machines are studied in computer science but seldom done in numerical analysis, see [4, 24, 14]. The pioneering work about the quantum complexity for numerical problem was done by Novak, [19]. After that, a series of papers about summation of sequences and multivariate integration of functions by Novak and Heinrich were published, see [12, 10, 11]. In [25], Traub initially discussed the quantum complexity of path integration.

In this paper, we continue the study of the problem of high-dimensional integration. Usually, the need to understand the complexity of the problems in the deterministic and randomized settings will help to judge the possible gains by quantum computation. In information-based complexity theory, the complexity of integration problems is well known for classical function classes. Recently, Fang and Ye [7] obtained the exact order of integration problem for anisotropic Sobolev classes and Holder-Nikolskii classes in the classical deterministic and randomized settings. Our goal is to study the complexity in the quantum computation model. Compared to the known results of complexities for some anisotropic classes, we hope that there exists an essential speed-ups under quantum computation similar to what happens for the classical Sobolev classes.

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We obtain the optimal order of high-dimensional integration complexity in the quantum computation model for anisotropic Sobolev classes $W^{\mathbf{r}}_{\infty}([0,1]^d)$ and Hölder Nikolskii classes $H^{\mathbf{r}}_{\infty}([0,1]^d)$. Our method is based on the discrete skill which is used in [11]. But we develop some new skills to overcome the difficulties of anisotropy and weaker smoothness which arise from the the study of our classes. For more details on the quantum setting for numerical problems we refer to [10]. For general background on quantum computing we refer to the surveys [8, 21] and to the monographs [16, 22].

We organize this paper as follows. In section two, we review the quantum computation model. In section three, the integration problems in anisotropic classes are briefly introduced. Moreover, we present the main results of our paper. Section four reviews some known results which is used in the proof of theorems. Finally, the proof of the new results are presented in section five.

2. Quantum Computation Model

In this section we introduce the quantum computation model. We start with adopting some notations following [11, 19]. For nonempty sets Ω and K we denote the set of all function from Ω to K by $\mathcal{F}(\Omega, K)$. Let G be a normed space with scalar field $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$, and let S be any mapping from F to G, where $F \subset \mathcal{F}(\Omega, \mathbb{R})$. we want to approximate $S(f)$ for $f \in F$ by quantum computations. Denote

$$\mathbb{Z}[0, N) := \{0, \dots, N-1\}$$

for $N \in \mathbb{N}$. Let H_m be m-fold tensor product of H_1, two-dimensional Hilbert space over $\mathbb{C}$, and let $\{e_0, e_1\}$ be two orthonormal basis of H_1. An orthonormal basis of H_m, denoted by $\mathcal{C}_m$, consist of the vectors $|l> := e_{i_0} \otimes \dots \otimes e_{i_m}$ $(l \in \mathbb{Z}[0, 2^{m-1}))$, where $\otimes$ is the tensor product, $i_j \in \{0,1\}$ and $l = \sum_{j=0}^{2^m-1} i_j 2^{m-1-j}$. Let $\mathcal{U}(H_m)$ stand for the set of unitary operator on H_m.

Two mappings are defined respectively by

$$\tau : Z \to \Omega \qquad \text{and} \qquad \beta : K \to \mathbb{Z}[0, 2^{m''}).$$

where for m, m', $m'' \in \mathbb{N}$, $m' + m'' \le m$ and Z is the nonempty subset of $\mathbb{Z}[0, 2^{m'})$. A quantum query on F is give by a tuple

$$Q = (m, m', m'', Z, \tau, \beta),$$

and the number of quits $m(Q) := m$. We define the unitary operator Q_f for a given query Q by setting for each $f \in F$

$$Q_f|i>|x>|y> := \begin{cases} |i>|x \oplus \beta(f(\tau(i)))>|y> & \text{if } i \in Z, \\ |i>|x>|y> & \text{otherwise,} \end{cases}$$

where set $|i>|x>|y> \in \mathcal{C}_m := \mathcal{C}_{m'} \otimes \mathcal{C}_{m''} \otimes \mathcal{C}_{m-m'-m''}$ and denote addition modulo $2^{m''}$ by $\oplus$.

Let tuple $A = (Q, (U_j)_{j=0}^n)$ denote a quantum algorithm on F with no measurement, where Q is a quantum query on F, $n \in \mathbb{N}_0$ $(\mathbb{N} = \mathbb{N} \bigcup \{0\})$ and $U_j \in \mathcal{U}(H_m)$, with $m = m(Q)$. For each $f \in F$, we have $A_f \in \mathcal{U}(H_m)$ with the following form

$$A_f = U_n Q_f U_{n-1} \dots U_1 Q_f U_0.$$