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ON COEFFICIENT POLYNOMIALS OF CUBIC HERMITE-PADÉ APPROXIMATIONS TO THE EXPONENTIAL FUNCTION *1)

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Abstract

The polynomials related with cubic Hermite-Padé approximation to the exponential function are investigated which have degrees at most n, m, s respectively. A connection is given between the coefficients of each of the polynomials and certain hypergeometric functions, which leads to a simple expression for a polynomial in a special case. Contour integral representations of the polynomials are given. By using of the saddle point method the exact asymptotics of the polynomials are derived as n, m, s tend to infinity through certain ray sequence. Some further uniform asymptotic aspects of the polynomials are also discussed.

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Key words: Padé-type approximant, Cubic Hermite-Padé approximation, Hypergeometric function, Saddle point method.

1. Introduction

Hermite-Padé approximation to the exponential function was introduced by Hermite [5] who considered expressions of the form

$$t_k(x)e^{s_k x} + t_{k-1}(x)e^{s_{k-1}x} + \cdots + t_1(x)e^{s_1 x} = O(x^h), \tag{1.1}$$

where $t_1(x), t_2(x), \cdots, t_k(x)$ are polynomials, of specified degrees,chosen so that h is as large as possible.

Included, of course, in expressions of type (1.1) are both the ordinary Padé approximations

$$\hat{P}_n(x)e^{-x} + \hat{Q}_m(x) = O(x^{n+m+1}) \tag{1.2}$$

with deg($\hat{P}_n$)$\leq n$, deg($\hat{Q}_m$)$\leq m$, $\hat{P}_n(0) \neq 0$, and the quadratic Hermite-Padé approximations [3,4]

$$\tilde{P}_n(x)e^{-2x} + \tilde{Q}_m(x)e^{-x} + \tilde{R}_s(x) = O(x^{n+m+s+2}) \tag{1.3}$$

with deg($\tilde{P}_n$)$\leq n$,deg($\tilde{Q}_m$)$\leq m$,deg($\tilde{R}_s$)$\leq s$, $\tilde{P}_n(0) \neq 0$.

In this paper, we wish to investigate a number of properties of the polynomials P_n, T_l, Q_m and R_s that arise from the solution of the following cubic Hermite-Padé approximations

$$P_n(x)e^{-3x} + T_l(x)e^{-2x} + Q_m(x)e^{-x} + R_s(x) = O(x^{n+m+s+l+3}), \tag{1.4}$$

with deg(P_n)$\leq n$, deg(T_l)$\leq l$, deg(Q_m)$\leq m$, deg(R_s)$\leq s$, P_n monic. But as is well known, if we set $x = y - \frac{a}{3}$, then any cubic equation $x^3 + ax^2 + bx + c = 0$ can be transformed into the following form

$$y^3 + (b - \frac{a^2}{3})y + (\frac{2}{27}a^3 - \frac{1}{3}ab + c) = 0.$$

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So without loss of generality, in this paper we only consider approximations to e^{-x} generated by finding polynomials P_n, Q_m and R_s so that

$$E_{nms}(x) := P_n(x)e^{-3x} + Q_m(x)e^{-x} + R_s(x) = O(x^{n+m+s+2}). \tag{1.5}$$

The explicit formulae for these unique polynomials are known; in the super-diagonal case $n = m = s$, they were obtained by Wang & Zheng [12] and for arbitrary $n, m, s \in \mathbf{N}$, they can be found in Zheng & Wang [13].

2. The Polynomials P_n, Q_m and R_s

The polynomials P_n, Q_m and R_s with deg(P_n)= n, deg(Q_m)= m, deg(R_s)= s, P_n monic, that satisfy (1.5) are given by (cf. Zheng & Wang [13])

$$P_n(x) = n! \sum_{j=0}^{n} \frac{p_j x^j}{j!}. \tag{2.1}$$

where, for $0 \le j \le n$,

$$p_j = 2^{j-n} \sum_{k=0}^{n-j} \left(\frac{2}{3}\right)^k \binom{n+m-k-j}{m} \binom{s+k}{s}; \tag{2.2}$$

$$Q_m(x) = -\frac{3^{s+1}}{2^n} n! \sum_{j=0}^{m} \frac{q_j x^j}{j!}, \tag{2.3}$$

where, for $0 \le j \le m$,

$$q_j = \sum_{k=0}^{m-j} (-2)^{k+j} \binom{n+m-k-j}{n} \binom{s+k}{s}; \tag{2.4}$$

$$R_s(x) = (-1)^m 2^{m+1} 3^{s-n} n! \sum_{j=0}^{s} \frac{r_j x^j}{j!}, \tag{2.5}$$

where, for $0 \le j \le s$,

$$r_j = (-1)^j \sum_{k=0}^{s-j} \frac{1}{3^k} \binom{s+m-k-j}{m} \binom{n+k}{n}. \tag{2.6}$$

We observe that each of the polynomials P_n, Q_m, and R_s depends on all three positive integers n, m, and s and the subscript merely denotes the degree of the polynomial in each case. Writing $P_n(x) = P(n, m, s; x), Q_m(x) = Q(n, m, s; x)$,and $R_s(x) = R(n, m, s; x)$.

Our first result establishes a connection between the coefficients of P_n, Q_m, R_s and certain ${}_2F_1$ hypergeometric functions.We recall the definition of the Gauss function (cf.[1])

$${}_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \tag{2.7}$$

where

$$(\alpha)_k := \begin{cases} \alpha(\alpha+1)\cdots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha), & \text{if } \ k \ge 1, \\ 1, & \text{if } \alpha \ne 0, k = 0. \end{cases} \tag{2.8}$$

If $t \in \mathbf{N}$,it follows immediately from (2.8) that

$$(-t)_k = \begin{cases} (-1)^k t!/(t-k)!, & \text{for } \ 0 \le k \le t, \\ 0, & \text{for } k > t. \end{cases} \tag{2.9}$$