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A MIXED FINITE ELEMENT METHOD FOR THE CONTACT PROBLEM IN ELASTICITY *

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Abstract

Based on the analysis of [7] and [10], we present the mixed finite element approximation of the variational inequality resulting from the contact problem in elasticity. The convergence rate of the stress and displacement field are both improved from $\mathcal{O}(h^{3/4})$ to quasi-optimal $\mathcal{O}(h|logh|^{1/4})$. If stronger but reasonable regularity is available, the convergence rate can be optimal $\mathcal{O}(h)$.

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Key words: Contact problem, Mixed finite element method.

1. Introduction

Variational inequalities arise mainly from the application of mechanics and physics, such as obstacle problem, unilateral problems, contact mechanics. The contact problem in elasticity is one of the mostly used models in the theory of variational inequality(see [6],[8]). Kikuchi and Oden[7] made a detailed analysis of the contact problem in elasticity with the mathematical model and numerical implementation of the models. Wang[10] improved the duality methods in the mixed finite element approximation. In this paper, we make an improvement of the error estimates from $\mathcal{O}(h^{3/4})$ to quasi-optimal $\mathcal{O}(h|logh|^{1/4})$. Under stronger but reasonable regularity assumption, the convergence rate can be optimal $\mathcal{O}(h)$.

Throughout the paper all the notation is followed with that in [7]. The notation of Sobolev spaces and the corresponding semi-norms, norms is taken from [1]. In addition, the frequently used constant C is a generic positive constant whose value may be different under different context. Bold Latin letters like $\boldsymbol{u}, \boldsymbol{v}$ represent for vector quantities and the summation convention of repeated indices over $1, 2$ is adopted. The paper is organized as follows: In section 2, we introduce some notation and present the framework of the contact problem. In section 3, mixed problem is derived and its finite element approximation is given. Finally we show our main results and the proofs in section 4.

2. The Framework of the Contact Problem

The contact problem in elasticity arises from deformable solid mechanics. Suppose $\Omega \subset \mathbb{R}^2$ is a Lipschitz bounded domain, and its boundary $\partial\Omega$ consists of three non-overlapping parts Γ_D, Γ_C and Γ_F. The displacement field $\boldsymbol{u}$ of Ω is fixed along Γ_D (Dirichlet condition) with $\text{meas}(\Gamma_D) > 0$ while Γ_C is the contact region subjected to a frictionless foundation. Moreover,

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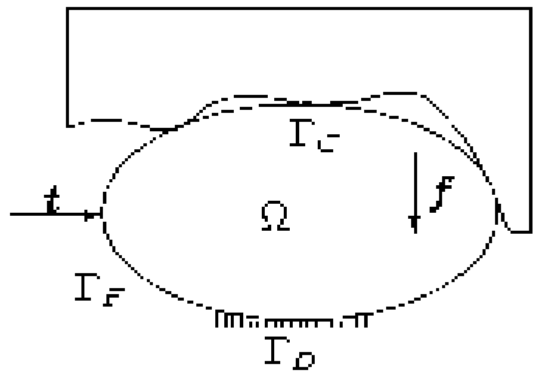

Fig 1.1

Γ_C and Γ_D are not adjacent, and Γ_F is the "glacis" between them with Neumann condition, i.e., the suface traction force $\boldsymbol{t}$ is applied to Γ_F. The body force is denoted by $\boldsymbol{f}$, and $g \in H_{00}^{1/2}(\Gamma_C \cup \Gamma_F)$(see Fig 1.1).

The general continuous setting of the contact problem in elasticity in $\mathbb{R}^2$ can be illustrated as the following mathematical model: to find the displacement field $\boldsymbol{u} \in K$,

$$K = \{\boldsymbol{v} \in \boldsymbol{H^1_{\Gamma_D}(\Omega)} = (H^1_{\Gamma_D}(\Omega))^2 : \boldsymbol{v} \cdot \boldsymbol{n} = v_n \leq g \quad \text{on} \quad \Gamma_C\},$$

such that

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) \geq f(\boldsymbol{v} - \boldsymbol{u}), \qquad \forall \boldsymbol{v} \in K, \tag{2.1}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \sigma_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v})dx, \tag{2.2}$$

$$f(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx + \int_{\Gamma_F} \boldsymbol{t} \cdot \boldsymbol{v} ds. \tag{2.3}$$

The notation $H^1_{\Gamma_D}$ stands for the set of functions in $H^1(\Omega)$ which vanish on Γ_D. Besides, $\boldsymbol{\epsilon}$ with $\epsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$ denotes the linearized strain tensor field induced by a displacement field $\boldsymbol{v}$ and $\sigma_{ij}(\boldsymbol{v}) = E_{ijkl}\epsilon_{kl}(\boldsymbol{v})$ is the stress tensor with $E = (E_{ijkl})$ denoting the Hooke's tensor of the elastic material. Moreover, the Hooke's tensor E has the following properties :

$$\begin{cases} E_{ijkl} \in L^\infty(\Omega), \quad \|E_{ijkl}\|_{L^\infty} \leq M, \\ E_{ijkl} = E_{klij} = E_{jilk}, \\ E_{ijkl}(\boldsymbol{x})\xi_{ij}\xi_{kl} \geq m\xi_{ij}\xi_{ij}, \quad \text{ for all } \boldsymbol{x} \in \Omega,\, \boldsymbol{\xi} = (\xi_{ij}) \in S^2, \end{cases} \tag{2.4}$$

where S^2 denotes the set of the real symmetric matrices of order two. Furthermore, $E = (E_{ijkl})$ is invertible, and its inverse denoted by $C = (C_{ijkl})$ also satisfies similar properties:

$$\begin{cases} C_{ijkl} \in L^\infty(\Omega), \quad \|C_{ijkl}\|_{L^\infty} \leq m_1, \\ C_{ijkl} = C_{klij} = C_{jilk}, \\ C_{ijkl}(\boldsymbol{x})\tau_{ij}\tau_{kl} \geq M_1\tau_{ij}\tau_{ij}, \quad \text{ for all } \boldsymbol{x} \in \Omega,\, \boldsymbol{\tau} = (\tau_{ij}) \in S^2, \end{cases} \tag{2.5}$$