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AN EXPANDED CHARACTERISTIC-MIXED FINITE ELEMENT METHOD FOR A CONVECTION-DOMINATED TRANSPORT PROBLEM [*1)]

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Abstract

In this paper, we propose an Expanded Characteristic-mixed Finite Element Method for approximating the solution to a convection dominated transport problem. The method is a combination of characteristic approximation to handle the convection part in time and an expanded mixed finite element spatial approximation to deal with the diffusion part. The scheme is stable since fluid is transported along the approximate characteristics on the discrete level. At the same time it expands the standard mixed finite element method in the sense that three variables are explicitly treated: the scalar unknown, its gradient, and its flux. Our analysis shows the method approximates the scalar unknown, its gradient, and its flux optimally and simultaneously. We also show this scheme has much smaller time-truncation errors than those of standard methods. A numerical example is presented to show that the scheme is of high performance.

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Key words: Convection diffusion problems, Expanded characteristic mixed finite element method, Optimal error estimates, Numerical test.

1. Introduction

Given an open bounded domain $\Omega \subset R^2$ with a smooth boundary Γ and a time interval $(0,T]$, we consider the following convection-diffusion equation

$$\begin{cases} (a) & \frac{\partial c}{\partial t} + u(x)\cdot\nabla c - \nabla\cdot(a(x)\nabla c) = f(x,t), & in\ \Omega\times(0,T), \\ (b) & c(x,0) = c_0(x), & in\ \ \Omega, \end{cases} \tag{1}$$

where
1) c(x,t) denotes, for example, the concentration of a possible substance;
2) u(x) represents the velocity of the flow;
3) ∇ and $\nabla\cdot$ denote the gradient and the divergence operators respectively;
4) a(x) is sufficiently smooth and there exist constants a_1 and a_2 such that

$$0 < a_1 \le a(x) \le a_2 < +\infty; \tag{2}$$

5) f denotes a source term.

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This equation governs such phenomena as the flow of heat within a moving fluid, the transport of dissolved nutrients or contaminants within the groundwater, and the transport of a surfactant or tracer within an incompressible oil in a petroleum reservoir.

Because of molecular diffusion, a(x) is uniformly positive. Although this implies that the equation is uniformly parabolic, in many applications the Peclet number is quite high. Thus convection dominates diffusion, the equation is nearly hyperbolic in nature. The concentration often develops sharp fronts that are nearly shocks.

It is well known that strictly parabolic discretization schemes applied to the problem do not work well when it is convection dominated. It is especially difficult to approximate well the sharp fronts and conserve the material or mass in the system.

Effective discretization schemes should concentrate on the hyperbolic nature of the equation. Many such schemes have been developed, such as the explicit method of characteristics, upstream-weighted finite difference schemes [2], the streamline diffusion method [4], the least-squares mixed finite element method [8], the modified method of characteristics-Galerkin finite element procedure (MMOC-Galerkin)[12,13,14,15].

In this paper we propose a mixed method, called the expanded characteristic mixed finite element method. It is similar to MMOC-Galerkin for convection dominated transport problems in that we approximate the hyperbolic part of the equation along the characteristics, we use, however, the expanded mixed finite element method ([1],[16]) to discretize the diffusion part. This formulation expands the standard mixed formulation in the sense that three variables are explicitly treated; i.e., the scalar unknown, its gradient and its flux(the coefficient times the gradient). It is suitable for the case where the coefficient of the differential equations is a small tensor and does not need to be inverted.

An outline of this paper is as follows. In Section 2, we define an approximation to the characteristics and the expanded characteristic-mixed finite element formulation of the problem. We give the proof of the existence and uniqueness of the discrete problem in Section 3. In Section 4, we give some Lemmas which are important to our error analysis. In Section 5, the optimal order estimates for $c - c_h, \sigma - \sigma_h, \lambda - \lambda_h$ in $L^2(\Omega)$ are presented. The last Section is devoted to a numerical example.

2. The Expanded Characteristic-mixed Finite Element Method

We begin this section by introducing some notations.

We denote by $W^{k,p}(S)$ the standard Sobolev space of k-differential functions in $L^p(S)$. Let $\|\cdot\|_{k,p,S}$ be its norm and $\|\cdot\|_{k,S}$ be the norm of $H^k(S) = W^{k,2}(S)$ or $H^k(S)^2$, where we omit S if $S = \Omega$. When $k = 0$, we let $L^2(\Omega)$ denote the corresponding space defined on Ω, its norm written as $\|\cdot\|$.

We also use the following spaces that incorporate time dependence. Let $[a, b] \subset [0, T]$, X be a Sobolev space, and $f(x, t)$ be suitably smooth on $\Omega \times [a, b]$. Also, we define $L^p(a, b; X)$ and $\|f\|_{L^p(a,b;X)}$ as follows

$$L^p(a, b; X) = \{f : \int_a^b \|f(\cdot, t)\|_X^p dt < \infty\},$$

$$\|f\|_{L^p(a,b;X)} = (\int_a^b \|f(\cdot, t)\|_X^p dt)^{\frac{1}{p}},$$

where if $p = \infty$, the integral is replaced by the essential supreme.

In this paper we assume that $u(x)$ satisfies

$$|u(x)| + |\nabla \cdot u(x)| \leq K, \qquad \forall x \in \Omega, \tag{3}$$

here and throughout this paper K denotes different constants in different places.