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AN ITERATIVE HYBRIDIZED MIXED FINITE ELEMENT METHOD FOR ELLIPTIC INTERFACE PROBLEMS WITH STRONGLY DISCONTINUOUS COEFFICIENTS*1)

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Abstract

An iterative algorithm is proposed and analyzed based on a hybridized mixed finite element method for numerically solving two-phase generalized Stefan interface problems with strongly discontinuous solutions, conormal derivatives, and coefficients. This algorithm iteratively solves small problems for each single phase with good accuracy and exchange information at the interface to advance the iteration until convergence, following the idea of Schwarz Alternating Methods. Error estimates are derived to show that this algorithm always converges provided that relaxation parameters are suitably chosen. Numeric experiments with matching and non-matching grids at the interface from different phases are performed to show the accuracy of the method for capturing discontinuities in the solutions and coefficients. In contrast to standard numerical methods, the accuracy of our method does not seem to deteriorate as the coefficient discontinuity increases.

Key words: Mixed finite element method, Interface problems, Discontinuous solutions.

1. Introduction

Interface problems occur in many physical applications. Below is a description of alloy solidification [10, 11, 12, 9] that shows the importance and characteristics of interface problems.

In alloy solidification problems, the melting temperature is not known in advance, which is different from classical Stefan problems such as ice-melting in water. The melting temperature depends on the composition of the alloy. Typically, an alloy is considered to comprise a pure substance containing a small concentration of one or more secondary substances, called impurities. The solidification of an alloy calls for a simultaneous study of the processes of heat flow and the diffusion of impurities. We now describe the mathematical model of a simple two-phase alloy solidification process in one, two or three space dimensions, with x denoting the space coordinate vector [10, 12, 11, 9]. Let $\Omega_1(t)$ denote the solid (alloy) region and $\Omega_2(t)$ the liquid (impurity) region, which are separated by the interface denoted by $\Gamma(t)$, where t represents time. Note the solid and liquid regions and the interface change with time t. Let u_1, c_1, K_1, and D_1 be the temperature, concentration of impurity, heat conductivity, and mass diffusion coefficient, respectively, in the solid region $\Omega_1(t)$, and u_2, c_2, K_2, and D_2 be the corresponding quantities in the liquid region $\Omega_2(t)$; see Figure 1.1. Then the partial differential equations modeling the process can be expressed as:

$$\rho_1 \frac{\partial u_1}{\partial t} = \nabla \cdot (K_1 \nabla u_1) + q_1, \quad \frac{\partial c_1}{\partial t} = \nabla \cdot (D_1 \nabla c_1), \quad \text{in } \Omega_1(t), \tag{1}$$

$$\rho_2 \frac{\partial u_2}{\partial t} = \nabla \cdot (K_2 \nabla u_2) + q_2, \quad \frac{\partial c_2}{\partial t} = \nabla \cdot (D_2 \nabla c_2), \quad \text{in } \Omega_2(t), \tag{2}$$

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$$u_1 = u_2, \quad K_2 \frac{\partial u_2}{\partial \nu} - K_1 \frac{\partial u_1}{\partial \nu} = -L v_\nu, \quad \text{on } \Gamma(t), \tag{3}$$

$$c_1 = c_S(u_1),\ c_2 = c_L(u_2), \quad D_2 \frac{\partial c_2}{\partial \nu} - D_1 \frac{\partial c_1}{\partial \nu} = (c_1 - c_2) v_\nu, \quad \text{on } \Gamma(t), \tag{4}$$

together with appropriate boundary and initial conditions, where v_ν denotes the speed at which the interface is moving along its normal direction ν, $q_1,\ q_2$ and $\rho_1,\ \rho_2$ are sources or sinks and specific heat in $\Omega_1,\ \Omega_2$, respectively, L is the latent heat. Equations (3) and (4) are the so-called generalized Stefan conditions.

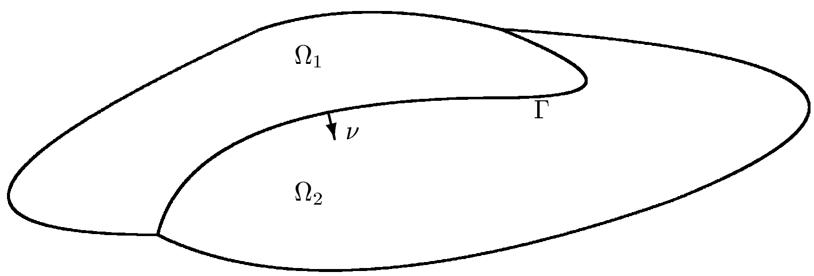

Figure 1.1: Two-phase alloy solidification with a moving interface $\Gamma(t)$ between the solid region $\Omega_1(t)$ and the liquid region $\Omega_2(t)$. Concentration of impurity is discontinuous across the interface.

Another example is multiphase immiscible flow of incompressible fluids with different densities and viscosities and surface tension. The governing equations in each fluid are the Navier-Stokes equations. The effect of surface tension is to balance the jump of the normal stress along the fluid interface, which gives rise to a free boundary condition for the discontinuity of the normal stress across the interface of the fluids. In the case of inviscid flows, the above jump condition is reduced to a discontinuity in pressure across the interface proportional to the curvature.

Interface problems such as the two mentioned above are difficult to solve by using conventional numerical methods since the the coefficients in different phases can be strongly discontinuous across the interface. Standard numerical methods such as finite element and mixed finite element algorithms are mainly designed to deal with problems with continuous or moderately discontinuous coefficients. For problems with strongly discontinuous coefficients, their accuracy can become arbitrarily inaccurate; see [31, 24] for some explanations and numerical examples. In particular, Nielsen [24] gave an example using standard finite element method whose accuracy deteriorates from 0.0044 to 0.0290 (or the error increased 559 percent) when the coefficient jump increases from 2 to 16. Vavasis [31] gave examples on which standard finite element methods fail on current computers. Note that modern preconditioners based on domain decomposition and multigrid cannot expect to improve the accuracy, although they may dramatically improve the efficiency of the solution process.

On the other hand, discontinuities in the solution or its normal derivatives can also present another difficulty; see [19, 20, 18, 32]. Note that the standard finite element theory [7] requires the solution be continuous (in $H^1(\Omega)$) and the mixed finite element theory [4] requires the normal derivative of the solution be continuous. Here we face a class of problems whose solution and its normal derivative can be discontinuous in the physical domain. Thus standard finite