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ON THE CONVERGENCE OF IMPLICIT DIFFERENCE SCHEMES FOR HYPERBOLIC CONSERVATION LAWS*1)

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Abstract

This paper is to treat implicit difference approximations to hyperbolic conservation laws with non-convex flux. The convergence of the approximate solution toward the entropy solution is established for the general weighted implicit difference schemes, which include some well-known implicit and explicit difference schemes.

Key words: Conservation laws, weighted implicit schemes, entropy solution.

1. Introduction

We are are interested in the following Cauchy problem for scalar conservation laws

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = 0, \ (x,t) \in \mathcal{R} \times \mathcal{R}^+, \\ u(0,x) = u_0(x), \ x \in \mathcal{R}, \end{cases} \tag{1.1}$$

where the initial data $u_0 \in BV(\mathcal{R})$ and the flux function $f \in C^2(\mathcal{R})$.

It is well known that this problem may not always have a smooth global solution even if the initial data u_0 is adequate smooth [11-14]. Thus, we consider its weak solution so that the problem (1.1) might have a global solution allowing discontinuities, e.g. shock wave etc.

A weak solution to the problem (1.1) is a function $u \in L^\infty(\mathcal{R} \times \mathcal{R}^+)$ satisfying:

$$\iint_{\mathcal{R}\times\mathcal{R}^+} \left(u\varphi_t + f(u)\varphi_x\right) \, dxdt + \int_{\mathcal{R}} u_0(x)\varphi(x,0) \, dx = 0, \tag{1.2}$$

for all $\varphi \in C^1(\mathcal{R} \times \mathcal{R}^+)$, with compact support in $\mathcal{R} \times \mathcal{R}^+$.

On the other hand, the week solutions to problem (1.1) are not necessarily unique. A physically relevant solution (also called the entropy solution) is characterized by a entropy condition

$$\iint_{\mathcal{R}\times\mathcal{R}^+} \left(U(u)\varphi_t + F(u)\varphi_x\right) \, dxdt \le 0, \tag{1.3}$$

for all positive test functions $\varphi \in C^1(\mathcal{R} \times \mathcal{R}^+)$, with compact support in $\mathcal{R} \times \mathcal{R}^+$, where the entropy function $U(u) \in C^2(\mathcal{R})$ is convex, i.e. $U''(u) > 0$, and the entropy flux function $F(u)$ satisfies

$$F'(u) = U'(u)f'(u).$$

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Numerical methods have been derived to approximate conservation laws in (1.1) in last two decades years, and have been applied to numerical simulations of many problems appeared in science and engineering. However, studies of convergence of approximate solutions to the entropy solution are still open for most of these numerical methods. Although there exist some related works (see references [1,2,15-23]), either some quantities depending on the space mesh size are always introduced in their investigating processes, or convex flux is only considered. In general, the difference schemes only depend on the ratio of the mesh sizes. Moreover, the introduction of these quantities may be improper for practical applications.

In this paper we study the convergence of the approximate solutions of (1.1) obtained by a class of weighted implicit schemes. They include some well-known implicit or explicit difference schemes for hyperbolic conservation laws with non-convex flux.

The paper is organized as follows. In section 2, we present construction of general weighted implicit difference schemes for one-dimensional scalar conservation laws in different form. In section 3, the convergence of the general weighted implicit difference schemes is study. Finally we give some remarks in the last section.

2. Weighted Implicit Difference Schemes

We consider the problem of construction of a general weighed implicit scheme (see [24]). For the sake of simplicity, the case of uniform grids is only considered. Let Δt and Δx be given positive numbers. Set $x_j = j\Delta x, x_{j+\frac{1}{2}} = \frac{1}{2}(x_j + x_{j+1})$ and $t^n = n\Delta t$. In the following, we also use some difference notations: $\Delta_- u_j = u_j - u_{j-1}$, $\Delta_+ u_j = u_{j+1} - u_j$, $\Delta_{j+\frac{1}{2}} u = u_{j+1} - u_j$, and $\delta_x^2 u_j = u_{j+1} - 2u_j + u_{j-1}$.

The approximate solution $u_\Delta(t,x) = u_j^n$ for $(t,x) \in [t^n, t^{n+1}) \times (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, is given by the following three-point weighted implicit difference scheme in conservative form

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{\Delta x}(\hat{h}_{j+\frac{1}{2}} - \hat{h}_{j-\frac{1}{2}}) = 0, \tag{2.1a}$$

where

$$\hat{h}_{j+\frac{1}{2}} = \theta h_{j+\frac{1}{2}}^{n+1} + (1-\theta) h_{j+\frac{1}{2}}^n, \ \ h_{j+\frac{1}{2}} = h(u_j, u_{j+1}), \tag{2.1b}$$

θ is positive parameter, $0 \le \theta \le 1$, and numerical flux function $h_{j+\frac{1}{2}}$ is Lipschitz continuous. We require the numerical flux function to be consistent with flux $f(u)$ in the following sense

$$h(u,u) = f(u), \tag{2.2}$$

and the initial data is projected onto the space of piecewise constant functions by the restriction

$$u_j^0 = \frac{1}{\Delta x}\int_{I_j} u_0(x)dx, \quad I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \tag{2.3}$$

for $j \in Z$.

We say that difference scheme (2.1) is consistent with entropy condition (1.3) in an inequality of the following kind is satisfied:

$$U_j^{n+1} \le U_j^n - \lambda(\hat{H}_{j+\frac{1}{2}} - \hat{H}_{j-\frac{1}{2}}), \quad \lambda = \frac{\Delta t}{\Delta x}, \tag{2.4}$$

where $U_j^n = U(u_j^n)$, and $\hat{H}(u,v)$ is a numerical entropy flux consistent with entropy flux $F(u)$, i.e.,

$$\hat{H}(u,u) = F(u).$$

Here and below, we also assume that numerical flux $h_{j+\frac{1}{2}}$ can be written in viscous form

$$h_{j+\frac{1}{2}} = \frac{1}{2}\big(f(u_j) + f(u_{j+1}) - \frac{1}{\lambda}Q_{j+\frac{1}{2}}(u_{j+1} - u_j)\big), \tag{2.5}$$

where $Q_{j+\frac{1}{2}} = Q(u_j, u_{j+1}; \lambda)$ is the coefficient of numerical viscosity. For example, for Lax–Friedrichs scheme, $Q_{j+\frac{1}{2}} = 1$; for well–known Lax–Wendroff scheme, $Q_{j+\frac{1}{2}} = (\lambda a_{j+\frac{1}{2}})^2$, etc.