Journal of Computational Mathematics, Vol.20, No.2, 2002, 185–196.

EXPANSION OF STEP-TRANSITION OPERATOR OF MULTI-STEP METHOD AND ITS APPLICATIONS (I)*[1)]

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Abstract

We expand the step-transition operator of any linear multi-step method with order $s \geq 2$ up to $O(\tau^{s+5})$. And through examples we show how much the perturbation of the step-transition operator caused by the error of initial value is.

Key words: Multi-step method, Step-transition operator, Expansion.

1. Expansion of Step-Transition Operator

For an ordinarily differential equation

$$\frac{d}{dt}Z = f(Z), \quad Z \in \mathbb{R}^p, \tag{1}$$

any compatible linear m-step difference scheme

$$\sum_{k=0}^{m} \alpha_k Z_k = \tau \sum_{k=0}^{m} \beta_k f(Z_k) \quad \left(\sum_{k=0}^{m} \beta_k \neq 0\right), \tag{2}$$

can be characterized by a step-transition operator G (also denoted by G^τ): $\mathbb{R}^p \to \mathbb{R}^p$ satisfying

$$\sum_{k=0}^{m} \alpha_k G^k = \tau \sum_{k=0}^{m} \beta_k f \circ G^k, \tag{3}$$

where G^k stands for k-time composition of G: $G \circ G \cdots \circ G$ (refer to [1,2,3,5,6,7]). This operator G^τ can be represented as a power series in τ with first term equal to *identity* I.

Thus, this operator completely characterizes the multi-step scheme as: $Z_1 = G(Z_0), \cdots, Z_m = G(Z_{m-1}) = G^m(Z_0), \cdots$. For the expansion of this operator G, we give the following theorem:

Theorem 1. *If scheme (2) is of order $s \geq 2$, then the step-transition operator decided by equation (3) has the following expansion:*

$$G(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^{s+1} A(Z) + \tau^{s+2} B(Z) + \tau^{s+3} C(Z) + \tau^{s+4} D(Z) + O(\tau^{s+5}), \tag{4}$$

where $Z^{[0]} = Z$, $Z^{[1]} = f(Z)$, $Z^{[k+1]} = \frac{\partial Z^{[k]}}{\partial Z} Z^{[1]}$ for $k = 1, 2, \cdots$. And

$$A = \lambda Z^{[s+1]}, \quad \lambda = \frac{\sum_{k=0}^{m} \left\{ \frac{k^s}{s!}\beta_k - \frac{k^{s+1}}{(s+1)!}\alpha_k \right\}}{\sum_{k=0}^{m} k\alpha_k}; \tag{4.1}$$

* Received January 11, 1999.
[1)]This research is supported by Special Funds for Major State Basic Research Projects of China (No. G1999032801-10 and No. G1999032804), and by the knowledge innovation program of the Chinese Academy of Sciences and a grant (No.19801034) from National Natural Science Foundation of China.

$$B = \mu Z^{[s+2]} + \frac{\lambda}{2} Z_z^{[1]} Z^{[s+1]}, \quad \mu = \frac{\sum_{k=0}^m \left[\frac{k^{s+1}}{(s+1)!}\beta_k - \frac{k^{s+2}}{(s+2)!}\alpha_k - \frac{k^2-k}{2}\lambda\alpha_k\right]}{\sum_{k=0}^m k\alpha_k}; \tag{4.2}$$

$$\begin{aligned} C =& \nu Z^{[s+3]} + \left(\rho + \frac{\lambda}{6}\right) Z_z^{[1]} Z_z^{[1]} Z^{[s+1]} \\ &+ \left(\rho - \frac{\lambda}{12} + \frac{\mu}{2}\right) Z_z^{[1]} Z^{[s+2]} + \left(2\rho + \frac{\lambda}{3}\right) Z_{z^2}^{[1]} Z^{[1]} Z^{[s+1]}, \\ \nu =& \frac{\sum_{k=0}^m \left[\frac{k^{s+2}}{(s+2)!}\beta_k - \frac{k^{s+3}}{(s+3)!}\alpha_k - \left(\frac{2k^3-3k^2+k}{12}\lambda + \frac{k^2-k}{2}\mu\right)\alpha_k\right]}{\sum_{k=0}^m k\alpha_k}, \\ \rho =& \frac{\sum_{k=0}^m \left[\frac{k^2}{2}\beta_k - \frac{k^3}{6}\alpha_k\right]}{\sum_{k=0}^m k\alpha_k}\lambda; \end{aligned} \tag{4.3}$$

$$\begin{aligned} D =& \xi Z^{[s+4]} + \left\{\sigma - \frac{\rho}{2} + \chi - \frac{\mu}{12} + \frac{\nu}{2} - \eta\right\} Z_z^{[1]} Z^{[s+3]} \\ &+ \left\{\sigma - \frac{\lambda}{24} + \chi + \frac{\mu}{6} - \eta\right\} Z_z^{[1]} Z_z^{[1]} Z^{[s+2]} \\ &+ \left\{3\sigma - \rho - \frac{\lambda}{24} + 2\chi + \frac{\mu}{3} - 3\eta\right\} Z_{z^2}^{[1]} Z^{[1]} Z^{[s+2]} \\ &+ \left\{\sigma + \frac{\rho}{2} + \frac{\lambda}{24} - \epsilon\zeta\right\} Z_z^{[1]} Z_z^{[1]} Z_z^{[1]} Z^{[s+1]} \\ &+ \left\{2\sigma + \rho + \frac{\lambda}{12} - \eta - \epsilon\zeta\right\} Z_z^{[1]} Z_{z^2}^{[1]} Z^{[1]} Z^{[s+1]} \\ &+ \left\{3\sigma + \frac{\lambda}{8} - \eta - 2\epsilon\zeta\right\} Z_{z^2}^{[1]} Z^{[1]} \left(Z_z^{[1]} Z^{[s+1]}\right) \\ &+ \left\{3\sigma + \frac{\lambda}{8} - 2\eta - \epsilon\zeta\right\} Z_{z^2}^{[1]} Z^{[2]} Z^{[s+1]} \\ &+ \left\{3\sigma + \frac{\lambda}{8} - 2\eta - \epsilon\zeta\right\} Z_{z^3}^{[1]} \left(Z^{[1]}\right)^2 Z^{[s+1]}, \\ \xi =& \frac{\sum_{k=0}^m \left\{\frac{k^{s+3}}{(s+3)!}\beta_k - \left[\frac{k^{s+4}}{(s+4)!} + \frac{k^4-2k^3+k^2}{24}\lambda + \frac{2k^3-3k^2+k}{12}\mu + \frac{k^2-k}{2}\nu\right]\alpha_k\right\}}{\sum_{k=0}^m k\alpha_k}, \\ \sigma =& \frac{\sum_{k=0}^m \left[\frac{k^3}{6}\beta_k - \frac{k^4}{24}\alpha_k\right]}{\sum_{k=0}^m k\alpha_k}\lambda, \\ \chi =& \frac{\sum_{k=0}^m \left[\frac{k^2}{2}\beta_k - \frac{k^3}{6}\alpha_k\right]}{\sum_{k=0}^m k\alpha_k}\mu, \\ \eta =& \frac{\sum_{k=0}^m \frac{k^2-k}{2}\alpha_k}{\sum_{k=0}^m k\alpha_k}\rho, \\ \zeta =& \frac{\sum_{k=0}^m \frac{k^2-k}{2}\alpha_k}{\sum_{k=0}^m k\alpha_k}\lambda^2, \\ \epsilon =& \begin{cases} 1, & when\ s = 2; \\ 0, & when\ s \geq 3. \end{cases} \end{aligned} \tag{4.4}$$